Path categories and inverse diagrams

Evio Paauw

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Abstract

In these notes we give outlines for a proof that every object in a path category is an internal Kan complex. We show that path categories are closed under (homotopical) inverse diagrams, and that the category of path categories is a fibration category. We moreover give the first steps in showing that the evaluation functor from homotopical semisimplicial objects in a path category to the path category itself, is a weak equivalence.

1 Introduction

As the name suggests, the main concept of homotopy type theory is to study the way that techniques from homotopy theory can be applied to type theory. The idea can be roughly described as follows. Given a type A, and two elements a and b of type A, one can consider a new type which is the ***identity type***; it is denoted as $\mathrm{Id}_A(a,b)$ and consists of elements which can be interpreted as 'proofs that a and b are the same'. That these identity types are not necessarily vacuous concepts and empty types, and more interestingly, give A the structure of a groupoid, was first shown by Hoffman and Streicher in [6]. A natural step next step is to consider the identity type of two elements α, β in $\mathrm{Id}_A(a,b)$. As the elements of $\mathrm{Id}_A(a,b)$ could be interpreted as paths between points a and b, the identity type $\mathrm{Id}_{\mathrm{Id}_A(a,b)}(\alpha,\beta)$ intuitively consists of homotopies between α and β. As we can inductively define higher identity types, we end up with the idea that a type A should have the structure of an ∞-groupoid. This idea that A has such a structure was proven in [13]. More precisely, van den Berg and Garner here show that the types, interpreted as objects in their associated syntactic category, are internal weak ∞-groupoids.

Over the past years, there have been various attempts to abstract the structure of a syntactic category to a broader, cleaner, pure categorical notion, capturing all of the relevant homotopical structures. Most of these structures look like fibration categories, as defined by Brown in [2], but with some extra axioms. Among them are the type-theoretic fibration categories, as defined by Shulman in [11], tribes, as defined by Joyal in [7], identity type categories as defined by van den Berg and Garner in [13], and path categories, as defined by van den Berg and Moerijk in [14]. That the syntactic categories of dependent type theories indeed satisfies these properties is mostly characterized by the existence of certain factorisation systems, as studied in [5].

A natural question one can ask is whether in these categorical structures, we also have that every object can be interpreted as an internal ∞-groupoid, giving a nice generalisation of van den Berg and Garner's result. It has been shown in [1] that this holds true for the identity type categories. An attempt to transfer this proof to path categories hasn't yet proven to be fruitful, as can be read in [9]. In this paper, we give outlines and first steps for another approach. The aim is to show that every object in some sense is weakly equivalent to an internal Kan complex. To do this, we first show that given a path category $\mathcal{C}$ and an inverse category I, a certain subcategory of the functor category $\mathcal{C}^I$ has the structure of a path category. In particular, we then have path category structure on the semisimplicial objects in $\mathcal{C}$. For the full subcategory consisting of all homotopical semisimplicial objects, that is, the category consisting of all semisimplicial objects in which every map is a weak equivalence, we want to show that its evaluation functor to $\mathcal{C}$ is a weak equivalence of path categories, that is, an equivalence of categories for the induced functor on their homotopy categories. This will show that we can picture every object in $\mathcal{C}$ as the bottom element of a homotopical semisimplicial object. Then, by taking its global sections, we obtain a semisimplicial set, for which we want to show that it has fillers for all horns.

2 Inverse diagrams on path categories

We will briefly give the definition of a path category for the sake of completeness, but we refer to [14] for all relevant results and proofs on path categories.

Definition 1. *A category $\mathcal{C}$ with two classes of maps, being the weak equivalences and fibrations, is called a* path categry *or* category with path objects*, if the following axioms are satisfied:*

- *i* *Fibrations are closed under composition.*
- *ii* *Pullbacks of fibrations exists and are fibrations again.*
- *iii* *Pullbacks of acyclic fibrations are acyclic fibrations.*
- *iv* *Weak equivalences satisfy* 2*-out-of-*6.
- *v* *Isomorphisms are acyclic fibrations and every acyclic fibration has a section.*
- *vi* *For every object X there is a path object PX.*
- *vii* *$\mathcal{C}$ has a terminal object and every map $X \to 1$ is a fibration.*

As mentioned in the introduction, we want to show that there is a canonical semisimplicial object which we can relate to any object in a path category, up to homotopy. To show that this is true we first want to show that path categories are closed under inverse categories. To do this, we almost mirror the proof given in [11] that type-theoretic fibration categories are closed under inverse diagrams. We also make use of ideas used in [4]. Let us first give some basic definitions on inverse categories.

Definition 2. *A category I is called an* inverse category *if the relation 'y receives a non-identity arrow from x' is well founded. Write $y \prec x$ for this relation. The* rank $\rho(x)$ *of an object x in I is defined as the supremum*

$$\sup_{y \prec x}(\rho(y)+1).$$

We define the ordinal rank[1] *of the category I as*

$$\sup_{x \in I}(\rho(x)+1).$$

[1]Mostly just abbreviated as 'rank', if the context is clear.

For now, we always write I for an inverse category and $\mathcal{C}$ for a path category, unless stated otherwise. For an object x in I we obtain a subcategory $x \mathbin{/\!\!/} I$ of the coslice category, excluding only the identity. This is itself a inverse category with ordinal rank equal to the rank of x.

We are interested in diagrams A of I in a path category $\mathcal{C}$. We only want to work with certain 'nice' diagrams of I in $\mathcal{C}$, for which appropriate limits exists in $\mathcal{C}$. This makes use of the notion of a *matching object*.

Definition 3. *Given a diagram A in $\mathcal{C}$ defined on the subcategory $\{y \mid y \prec x\}$, we write M_xA for the limit of the diagram, precomposed with the forgetful functor $x \mathbin{/\!\!/} I \to \{y \mid y \prec x\}$. It is called the* matching object.

We can extend a diagram which is defined on $\{y \mid y \prec x\}$ to a diagram on $\{y \mid y \preceq x\}$ by defining an object A_x and a map $A_x \to M_xA$. In order to study a path category structure on C^I, or an appropriate subcategory, we have to define what our fibrations will be. These are given by the so called *Reedy fibrations*.

Definition 4. *Let A and B be two diagrams of I in $\mathcal{C}$, and let $f : A \to B$ a natural transformation. We call f a* Reedy fibration *if A and B have all matching objects, and the map $A_x \to M_xA \times_{M_xB} B_x$, as in the following diagram:*

$$
\begin{tikzcd}
A_x \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& M_xA \times_{M_xB} B_x \arrow[r] \arrow[d] & B_x \arrow[d] \\
& M_xA \arrow[r] & M_xB
\end{tikzcd}
$$

is a fibration in $\mathcal{C}$.

These Reedy fibrations will be the fibrations in the path category structure. The weak equivalences will be the pointwise weak equivalences. With this notion of Reedy fibrations, a diagram A is fibrant if it has all matching objects, and all the maps $A_x \to M_xA$ are fibrations. We will often write a_x for this fibration $A_x \to M_xA$.

Definition 5. *A path category $\mathcal{C}$ has* Reedy I-limits *if for every Reedy fibrant A and B, and a morphism $f : A \to B$, we have that* *i* A *has a fibrant limit in* $\mathcal{C}$*,*

ii $\lim f$ *is a fibration if* f *is,*

iii $\lim f$ *is a weak equivalence if* f *is.*

We call I admissible *for* $\mathcal{C}$ *if* $\mathcal{C}$ *has* $(x \mathbin{/\!\!/} I)$*-limits for all* x *in* I*.*

We are now ready to state and prove our main theorem

Theorem 6. *Let* I *be admissible for* $\mathcal{C}$*. Then the full subcategory* $(C^I)_f$ *consisting of all Reedy fibrant diagrams in* C^I *has the structure of a path category.*

Proof. We will carefully check that all axioms of a path category are satisfied.

i *Fibrations are closed under composition.* Let $f : A \to B$ and $g : B \to C$ be Reedy fibrations. We want to show that $g \circ f$ is a Reedy fibration. Clearly, A and C have all matching objects. Each map $C_x \to M_x C$ is a fibration because C is fibrant. Hence, the pullbacks $M_x A \times_{M_x C} C_x$ exist. To show that the map from A_x to this pullback is a fibration, we consider the following diagram

$$\begin{tikzcd}
A_x \arrow[r] \arrow[dr] & M_xA \times_{M_xB} B_x \arrow[r] \arrow[d] & M_xA \times_{M_xC} C_x \arrow[r] \arrow[d] & M_xA \arrow[d] \\
& B_x \arrow[r] \arrow[dr] & M_xB \times_{M_xC} C_x \arrow[r] \arrow[d] & M_xB \arrow[d] \\
& & C_x \arrow[r] & M_xC
\end{tikzcd}$$

The three squares are all pullbacks by elementary theory on pullbacks. We now see that the map $A_x \to M_xA \times_{M_xC} C_x$ factorises as the map $A_x \to M_xA \times_{M_xB} B_x$, which is a fibration, followed by the map $M_xA \times_{M_xB} B_x \to M_xA \times_{M_xC} C_x$, which is the pullback of a fibration and hence a fibration.

ii *Pullbacks of fibrations exists and are fibrations again.* First, let us note that for a Reedy fibration $g : B \to C$, all its components are fibrations in $\mathcal{C}$. The map $B_x \to C_x$ is a composition of the fibration $B_x \to M_xB \times_{M_xC} C_x$ and the pullback of the map $M_xB \to M_xC$. This map $M_xB \to M_xC$ is $\lim_{x /\!\!/ I} f$, and hence a fibration, since $\mathcal{C}$ has all Reedy $(x \mathbin{/\!\!/} I)$-limits. Since every g_x is a fibration, we can conclude that the pullback of the fibration exists in $\mathcal{C}^I$, as pullbacks are computed pointwise.

For the pullback to be in $(C^I)_f$, it has to be fibrant. I.e, the map $(A \times_C B)_x \to M_x(A \times_C B) \cong M_x A \times_{M_x C} M_x B$ must be a fibration. To see this, we note that the map factorises through $A_x \times_{M_x C} M_x B$ as in the following diagram

$$\begin{tikzcd}
(A \times_C B)_x \cong A_x \times_{C_x} B_x \arrow[r] & A_x \times_{M_x C} M_x B \arrow[r] \arrow[d] & M_x A \times_{M_x C} M_x C \arrow[r] \arrow[d] & M_x B \arrow[d] \\
 & A_x \arrow[r] & M_x A \arrow[r] & M_x C
\end{tikzcd}$$

The map $A_x \times_{M_x C} M_x B \to M_x A \times_{M_x C} M_x B$ is a fibration as it is the pullback of $A_x \to M_x A$. The map $A_x \times_{C_x} B_x \to A_x \times_{M_x C} M_x B$ is a fibration too, as will become clear from the following diagram:

$$\begin{tikzcd}
A_x \times_{C_x} B_x \arrow[r] \arrow[d] & B_x \arrow[d] \arrow[dr] & \\
A_x \times_{M_x C} M_x B \arrow[r] \arrow[d] & M_x B \times_{M_x C} C_x \arrow[r] \arrow[d] & M_x B \arrow[d] \\
A_x \arrow[r] & C_x \arrow[r] & M_x C
\end{tikzcd}$$

We now want to show that the pullback of g is again a fibration. First note that $A \times_C B$ has all matching objects, as we have that

$$\lim_{x /\!/ I} (A \times_c B) \cong \lim_{x /\!/ I} A \times_{\lim_{x /\!/ I} C} \lim_{x /\!/ I} B \cong M_x A \times_{M_x C} M_x B,$$

where the last pullback exists as the map $M_x B \to M_x C$ is a fibration by an earlier remark. It remains to show that the map $(A \times_C B)_x \to M_x(A \times_C B) \times_{M_x A} A_x$ is a fibration. First consider the following diagram

$$\begin{tikzcd}
A_x \times_{C_x} B_x \arrow[r] \arrow[d] & B_x \arrow[d] \arrow[dr] & \\
A_x \times_{C_x} (M_x B \times_{M_x C} C_x) \arrow[r] \arrow[d] & M_x B \times_{M_x C} C_x \arrow[r] \arrow[d] & M_x B \arrow[d] \\
A_x \arrow[r] & C_x \arrow[r] & M_x C
\end{tikzcd}$$

All the squares are pullbacks. We claim that the map $A_x \times_{C_x} B_x \to A_x \times_{C_x} (M_x B \times_{M_x C} C_x)$ is the map we are looking for. First note that $A_x \times_{C_x} (M_x B \times_{M_x C} C_x)$ is isomorphic to $A_x \times_{M_x C} M_x B$. Now the following square shows that this is in fact isomorphic to $M_x(A \times_C B) \times_{M_x A} A_x$.

$$\begin{tikzcd}
(M_x A \times_{M_x C} M_x B) \times_{M_x A} A_x \arrow[r] \arrow[d] & A_x \arrow[d] \\
M_x A \times_{M_x C} M_x B \arrow[r] \arrow[d] & M_x A \arrow[d] \\
M_x B \arrow[r] & M_x C
\end{tikzcd}$$

iii *Pullbacks of acyclic fibrations are acyclic fibrations.* Immediate consequence of the previous proof and the fact that acyclic fibrations in $(C^I)_f$ are pointwise acyclic fibrations.

iv *Weak equivalences satisfy 2-out-of-6.* Immediate.

v *Isomorphisms are acyclic fibrations, and every acyclic fibration has a section.* Let $f : A \to B$ an isomorphism. In particular, all its components are acyclics and hence f is a weak equivalence and all its components are fibrations. Let us now show that it is a fibration. Because A and B are fibrant, they have all matching objects. Because an isomorphism is pulled back to an isomorphism, we have that $A_x \to M_x A \times_{M_x B} \times B_x \cong M_x A$ is a fibration as A is fibrant.

We now want to show the existence of a section. We will do this by induction on the rank of I.[2] The base case is clear, as for a discrete inverse diagram we can take just the pointwise sections. Now, let us assume that it holds for diagrams of rank n, and let I be a diagram of rank $n+1$. Consider the full subcategory J consisting of all the objects in I with rank lower than n. Let $f : A \to B$ be an acyclic fibration on the diagram I. In particular, it is an acyclic fibration between $A|_J$ and $B|_J$. Hence, we have a section $\tilde{g} : B|_J \to A|_J$. We now want to extend this section to the whole of B. Let x an element of rank n. For $\alpha : x \to y$, we write $l_{B,\alpha}$ for the map from $M_x B$

[2] Let us note that we only consider consider induction on natural numbers. Doing the transfinite case might be interesting too, but is not inside the scope of these notes.

to B_y, and b_x for the map $B_x \to M_x B$. Note that we have $l_{B,\alpha} \circ b_x = B(\alpha)$. We obtain a cone of $M_x B$ on the diagram induced by A on $x /\!\!/ I$, defined by $\tilde{g}_y \circ l_{B,\alpha}$ for each $\alpha : x \to y$. Because $\tilde{g}$ is natural by induction hypothesis, this is indeed a cone. By the universality of the limit we obtain a map $u_x : M_x B \to M_x A$, with the property that $l_{A,\alpha} \circ u_x = \tilde{g}_y \circ l_{B,\alpha}$. This u_x is the section of $(l_f)_x : M_x A \to M_x B$ due to the universality of $M_x B$. That is, if we consider the limiting cone $M_x B$, we see that

$$l_{B,\alpha} \circ (l_f)_x \circ u_x = f_y \circ l_{A,\alpha} \circ u_x = f_y \circ \tilde{g}_y \circ l_{B,\alpha} = l_{B,\alpha}$$

and hence $(l_f)_x \circ u_x = 1$. Now note that the map $\langle a_x, f_x \rangle$ is an acyclic fibration by basic reasoning. Hence, we have a section s_x of this map. We now claim that the map $g_x := s_x \circ \langle u_x \circ b_x, id \rangle$ gives us the desired section of f_x. From now on, we will write g_y for the previously used $\tilde{g}_y$. Let us show that g now is a natural transformation, and that it is indeed a section.

(a) *This defines a natural transformation* $g : B \to A$. For maps between two objects of rank lower than n we use that $\tilde{g}$ already was natural. Let us look at a map $\alpha : x \to y$. We want to show that the following diagram commutes

$$\begin{tikzcd} B_x \arrow[r, "B(\alpha)"] \arrow[d, "g_x"] & B_y \arrow[d, "g_y"] \\ A_x \arrow[r, "A(\alpha)"] & A_y \end{tikzcd}$$

Let us compute

$$\begin{aligned} A(\alpha) \circ g_x &= A(\alpha) \circ s_x \circ \langle u_x \circ b_x, id \rangle \\ &= l_{A,\alpha} \circ a_x \circ s_x \circ \langle u_x \circ b_x, id \rangle \\ &= l_{A,\alpha} \circ \pi_{M_x A} \circ \langle a_x, f_x \rangle \circ s_x \circ \langle u_x \circ b_x, id \rangle \\ &= l_{A,\alpha} \circ \pi_{M_x A} \circ \langle u_x \circ b_x, id \rangle \\ &= l_{A,\alpha} \circ u_x \circ b_x \\ &= g_y \circ l_{B,\alpha} \circ b_x \\ &= g_y \circ B(\alpha) \end{aligned}$$

(b) *This is a section of* f. We compute

$$\begin{aligned} f_x \circ g_x &= \pi_{B_x} \circ \langle a_x, f_x \rangle \circ g_x \\ &= \pi_{B_x} \circ \langle a_x, f_x \rangle \circ s_x \circ \langle u_x \circ b_x, id \rangle \\ &= \pi_{B_x} \circ \langle u_x \circ b_x, id \rangle \\ &= id. \end{aligned}$$

vi *For every object there is a path object.* Let $A : I \to \mathcal{C}$ in $(\mathcal{C}^I)_f$. We want to factorise $A \to A \times A$ as a weak equivalence followed by a fibration. Again we do this by induction. The base case is clear, as for a discrete category we can just take the path object factorisation induced by the one in $\mathcal{C}$. Now, let I be an inverse category of rank $n+1$, and let J as before. Let x of rank n. We can write PA^J for the path object of A restricted to J. Because we have all Reedy $(x \,/\!\!/\, I)$-limits, we have the limits $r_{M_xA} : M_xA \to M_xPA$ and $(s_{M_xA}, t_{M_xA}) : M_xPA \to M_xA \times M_xA$, which are respectively a weak equivalence and a fibration. So, we have a path object on M_xA which is the limit of the other path objects. We will now define PA_x, which will hopefully extend PA^J to a functor PA. Recall that we write a_x for the fibration $A_x \to M_xA$. We consider the following pullback

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & A_x \times A_x \arrow[d, "a_x \times a_x"] \\
PM_xA \arrow[r, "(s_{M_xA}, t_{M_xA})"'] & M_xA \times M_xA
\end{tikzcd}$$

We obtain a map from A_x to the pullback, $\langle r_{PM_xA} \circ a_x, \Delta_{A_x} \rangle$. We factorise this map as a weak equivalence followed by a fibration, and obtain a factorization of the diagonal Δ_{A_x} as summarized in the following diagram:

$$\begin{tikzcd}
A_x \arrow[dr, "r_x"] \arrow[ddrrr, bend left, "\Delta_{A_x}"] \arrow[dddr, bend right, "a_x"'] & & & \\
& PA_x \arrow[dr] & & \\
& & U \arrow[r] \arrow[d] & A_x \times A_x \arrow[d, "a_x \times a_x"] \\
& M_xA \arrow[r, "r_{M_xA}"'] & PM_xA \arrow[r, "(s_{M_xA}, t_{M_xA})"'] & M_xA \times M_xA
\end{tikzcd}$$

the map r_x is by definition a weak equivalence, and the resulting map $(s_x, t_x) : PA_x \to A_x \times A_x$ is a fibration since it is the composition of a fibration with the pullback of a fibration. The map $PA_x \to PM_xA$ which is needed for PA to be fibrant, is given by the obvious composition in the diagram, which is indeed a fibration since $a_x \times a_x$ is a fibration, and the map

$PA_x \to PM_xA$ is a composition of the fibration $PA_x \to U$ and the pullback of $a_x \times a_x$. We will denote this map by pa_x. It remains to show that this really extends PA^J to PA, and that r_A and (s_A, t_A) are well defined natural transformations. Note that we write r_A for the natural transformation $A \to PA$ as a whole, and r_x for its components. The map $PA(\alpha) : PA_x \to PA_y$ is the map from pa_x followed by the projection on PA_y corresponding to the map α.

(a) PA *is a funtor.* Let $\alpha : x \to y$ and $\beta : y \to z$. We want that $PA(\beta \circ \alpha) = PA(\beta) \circ PA(\alpha)$. This just follows from the properties of PM_xA being the limit.

(b) r_A *is a natural transformation.* Let $\alpha : x \to y$. We have to show that the following diagram commutes

$$\begin{tikzcd} A_x \arrow[r, "r_x"] \arrow[d, "A(\alpha)"] & PA_x \arrow[d, "PA(\alpha)"] \\ A_y \arrow[r, "r_y"] & PA_y \end{tikzcd}$$

Note that by definition of r_x and pa_x, the following diagram commutes

$$\begin{tikzcd} A_x \arrow[r, "r_x"] \arrow[d, "a_x"] & PA_x \arrow[d, "pa_x"] \\ M_xA \arrow[r, "r_{M_xA}"] & PM_xA \end{tikzcd}$$

The following square commutes by definition of r_{M_xA}.

$$\begin{tikzcd} M_xA \arrow[r, "r_{M_xA}"] \arrow[d, "\pi"] & PM_xA \arrow[d, "\pi"] \\ A_y \arrow[r, "r_y"] & PA_y \end{tikzcd}$$

with the vertical maps being the appropriate projections. Since $A(\alpha)$ is defined by composing a_x with this appropriate projection, and similar for $PA(\alpha)$, we obtain $r_y \circ A(\alpha) = PA(\alpha) \circ r_x$ by pasting these two commuting squares above together, or, writing π for the projection, we get

$$\begin{aligned} r_y \circ A(\alpha) &= r_y \circ \pi \circ a_x \\ &= \pi \circ r_{M_x A} \circ a_x \\ &= \pi \circ pa_x \circ r_x \\ &= PA(\alpha) \circ r_x \end{aligned}$$

(c) (s_A, t_A) *is a natural transformation.* This is a diagram chase very similar to the previous one.

vii *We have a terminal object, and every object is fibrant.* Immediate.

□

Whew! That was quite some work. The following lemma will ensure us that actually for many inverse categories, we indeed have that they are admissible for a path category $\mathcal{C}$.

Lemma 7. *Let I any finite inverse category. Then, the path category $\mathcal{C}$ has all Reedy I-limits.*

Proof. Exact copy of Lemma 11.8 in [11]. Because the construction of the limits will be useful in the discussion on the evaluation functor, we sketch the outlines. We do this by induction on the rank of I. The base case is an exact copy of Lemma 11.6 in [11]. Now suppose that it holds for finite inverse categories with rank n. Let I such that $\rho(I) = n+1$. Write J for the full subcategory of objects y with $\rho(y) < n$. Let A be a Reedy fibrant diagram in $\mathcal{C}^I$. Note that $A|_J$ is fibrant in $\mathcal{C}^J$. The limit $\lim_I A$ arises in the following pullback diagram:

$$\begin{tikzcd} \lim_I A \arrow[r] \arrow[d] & \prod_{\rho(x)=n} A_x \arrow[d] \\ \lim_J A|_J \arrow[r] & \prod_{\rho(x)=n} M_x A \end{tikzcd}$$

The limit in the left lower corner exists by the induction hypothesis, and $A|_J$ being fibrant. The map on the right is the product of fibrations and hence itself a fibration. Therefore, the pullback exists. □

Corollary 8. *The full subcategory $(\mathcal{C}^{\Delta_i^{\mathrm{op}}})_f$ of Reedy fibrant semisimplicial objects in $\mathcal{C}$, has the structure of a path category.*

As mentioned, we are interested in semisimplicial objects in which only weak equivalences play a role. They are called *homotopical diagrams.*

Lemma 9. *Given admissible I. The full subcategory $\mathcal{D} \subseteq (\mathcal{C}^I)_f$ of homotopical diagrams has the structure of a path category.*

Proof. First note that the terminal object is clearly homotopical. As we define the subcategory to be full, it is already clear that fibrations are closed under compositions, weak equivalences satisfy 2-out-of-6, every object is fibrant, isomorphisms are acyclic fibrations and acyclic fibrations have sections. It remains to show that pullbacks of fibrations exist, and that $\mathcal{D}$ has path objects. Or rather, that a pullback of a fibration is again homotopical. This follows precisely by Lemma 11.7 of Shulman, where the back face is the pullback diagram of $(A \times_C B)_x$, and the front is that of $(A \times_C B)_y$, for the map $\alpha : x \to y$ in I. Now for the path objects. Let PA be the path object of homotopical digram A in $(\mathcal{C}^I)_f$. We know that $r_x : A_x \to PA_x$ are all weak equivalences. Now for $\alpha : x \to y$ in I we have the naturality square

$$\begin{tikzcd}
A_x \arrow[r, "r_x"] \arrow[d, "A(\alpha)"] & PA_x \arrow[d, "PA(\alpha)"] \\
A_y \arrow[r, "r_y"] & PA_y
\end{tikzcd}$$

where we know that $A(\alpha), r_x$ and r_y are weak equivalences, and hence $PA(\alpha)$ is, by 2-out-of-3. □

3 The fibration category of path categories

In [8], it is shown that the category of all fibration categories has the structure of a fibration category for an appropriate notion of weak equivalence and fibration. We will do the same for an appropriate subcategory of all path categories, with a notion of weak equivalence which is the same as the one used in [8], and with a notion of fibration which is somewhere 'in between' the notions they use for fibrations between tribes and between fibration categories.

Definition 10. *Let $\mathcal{C}$ and $\mathcal{D}$ two path categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called* exact *if it preserves weak equivalences, fibrations, the terminal object and pullbacks along fibrations.*

Definition 11. *Let $\mathcal{C}$ and $\mathcal{D}$ two path categories. An exact functor $F : \mathcal{C} \to \mathcal{D}$ is an* equivalence of path categories *or a* weak equivalence *if the induced functor*

$$\tilde{F} : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$$

is an equivalence of categories in the ordinary sense.

There is a nice characterization of such weak equivalences given by Cisinski in [3].

Theorem 12. *Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor between path categories. Then F is a weak equivalence iff it reflects weak equivalences, and for every $f : Y \to FX$ in $\mathcal{D}$, we can find a map $u : X' \to X$ in $\mathcal{C}$, and weak equivalences $v : Y' \to Y$ and Y' to FX', such that the following diagram commutes:*

$$\begin{tikzcd} Y' \arrow[r, "w"] \arrow[d, "v"] & FX' \arrow[d, "Fu"] \\ Y \arrow[r, "f"] & FX \end{tikzcd}$$

This last property is also called the approximation property.

Proof. [3] □

Definition 13. *An exact functor $F : \mathcal{C} \to \mathcal{D}$ between path categories is a* fibration of path categories *if it satisfies the following properties:*

i It is an isofibration*: for every X in $\mathcal{C}$ and an isomorphism $f' : FX \to Y$, there is an isomorphism $f : X \to Y'$ such that $Ff = f'$.*

ii It has the lifting property for factorizations*: Given a morphism $f : X \to Y$ in $\mathcal{C}$, and a factorization $Ff = p' \circ i'$, with p a fibration and i a weak equivalence, there is a factorization $f = p \circ i$ such that $Fp = p'$ and $Fi = i'$.*

iii It has the lifting property for pseudofactorizations*: Given a morphism $f : X \to Y$ in $\mathcal{C}$ and a diagram*

$$\begin{tikzcd} FX \arrow[r, "Ff"] & FY \\ Z' \arrow[u, "i'"] \arrow[r, "s'"] & W' \arrow[u, "u'"'] \end{tikzcd}$$

where i' is an acyclic fibration, s' a weak equivalence and u' is a fibration, there exists a diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ Z \arrow[u, "i"] \arrow[r, "s"] & W \arrow[u, "u"'] \end{tikzcd}$$

such that $Fi' = i$ and $Fs' = s$ and $Fu' = u$.

iv It has the lifting property for acyclic fibrations*: if $f : X \to Y$ is an acyclic fibration in $\mathcal{C}$, and s' is a section Ff, then there is a section s of f such that $Fs = s'$.*

The following lemma, due to [12], gives a useful characterization of acyclic fibrations between path categories.

Lemma 14. *An exact functor $F : \mathcal{C} \to \mathcal{D}$ is an acyclic fibration iff the functor is a fibration, reflects weak equivalences and moreover it satisfies the following property: given a fibration $f : Y \to FX$ in $\mathcal{D}$, there is a fibration $f' : Y' \to X$ in $\mathcal{D}$ such that $Ff' = f$.*

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ satisfies the properties in the lemma. To show it is an acyclic fibration, we have to show that F has the approximation property. Let $f : Y \to FX$ a morphism in $\mathcal{D}$, factorize the map as

$$Y \xrightarrow{w} P \xrightarrow{p} FX$$

with p a fibration and w a weak equivalence. Then we have some fibration $p' : P' \to X$ such that $Fp' = p$. This now all fits in the following commuting diagram:

$$\begin{tikzcd} Y \arrow[r, "w"] \arrow[d] & FP' \arrow[d, "Fp'"] \\ Y \arrow[r, "f"] & FX \end{tikzcd}$$

which precisely is the diagram we want for the approximation property to hold, and hence F is an acyclic fibration.

Now suppose that F is an acyclic fibration, and let $f : Y \to FX$ a fibration in $\mathcal{D}$. We use the approximation property to get a diagram

$$\begin{tikzcd} Y' \arrow[r, "w"] \arrow[d, "v"] & FX' \arrow[d, "Fu"] \\ Y \arrow[r, "f"] & FX \end{tikzcd}$$

with v and w weak equivalences. The following pullback exists since f is a fibration

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_0"] & FX' \arrow[d, "Fu"] \\ Y \arrow[r, "f"] & FX \end{tikzcd}$$

Now factorize the map $\langle v, w\rangle : Y' \to P$ as

$$P \xrightarrow{i} Q \xrightarrow{g} P$$

with i a weak equivalence and g a fibration. We now have the following diagram

$$\begin{tikzcd} Q \arrow[r, "p_1\circ g"] \arrow[d, "p_0\circ g"] & FX' \arrow[d, "Fu"] \\ Y \arrow[r, "f"] & FX \end{tikzcd}$$

Note that since $p_0 \circ g \circ i = v$ and v and i being weak equivalences, the composite $p_0 \circ g$ is a weak equivalence by 2-out-of-3. By a similar argument $p_1 \circ g$ is a weak equivalence, and it is also a fibration since p_1 and g both are fibrations. Hence we have a diagram in d to which we can apply the lifting property for pseudofactorizations since F is a fibration, and in particular we obtain some $f' : Y' \to X$ such that $Ff' = f$. □

Theorem 15. *With the notions of weak equivalences and fibrations as above, the category **Pth** of path categories and exact functors, is a fibration category.*

Proof. We check the axioms.

i *Fibrations are closed under composition.* Immediate.

ii *The pullback of a fibration along any other map exists and is again a fibration.* Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{E} \to \mathcal{D}$ exact functors, and G a fibration of path categories. First we show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ has the structure of a path category, where the fibrations and weak equivalences both are defined pointwise. The only nontrivial axioms to check are the existence of path objects and the sections of acyclic fibrations. Let us first show that we have path objects. Let (X, Y) in the pullback, and let

$$X \xrightarrow{r_X} PX \xrightarrow{(s_X,t_X)} X \times X$$

a path object of X. Applying F gives a path object of FX, since F preserves weak equivalences, fibrations and products, being a pullback of fibrations. Since $FX = GY$, and $G(Y \times Y) = Y \times Y$, we can apply the lifting property for factorizations and obtain

$$Y \xrightarrow{r_Y} PY \xrightarrow{(s_Y,t_Y)} Y \times Y$$

such that $GPY = FPX$ and $Gr_Y = Fr_X$ and $G(s_Y, t_Y) = F(s_X, t_X)$. This gives us a path object

$$(X, Y) \xrightarrow{(r_X,r_y)} (PX, PY) \xrightarrow{((s_X,t_X),(s_Y,t_Y))} (X \times X, Y \times Y)$$

where we implicitly identify $(X, Y) \times (X, Y)$ with $(X \times X, Y \times Y)$, since limits commute with limits.

Let's now move to the section of acyclic fibrations. Let $(f, g) : (X, Y) \to (X', Y')$ an acyclic fibration. Then f is an acyclic fibration with a section s_f, and Fs_f is a section of Ff and hence also of Gg, as they are the same. By the lifting property of sections of acyclic fibrations, we obtain a section $s_g : Y \to Y'$ such that $Gs_g = Fs_f$, and hence (s_f, s_g) is a section of (f, g).

That the projection $p_0 : \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to \mathcal{C}$ is a fibration is also mostly routine. We give a short argument for the isofibration part, and leave the other three to the reader. If (X, Y) are in the pullback, and we have an isomorphism $i : X \to Z$ in $\mathcal{C}$. Then $Fi : FX \to FZ$ is an isomorphism, and hence $Fi : GY \to FZ$ is, as they are the same. Because G is a fibration we obtain $\tilde{i} : Y \to Z'$ such that $G\tilde{i} = Fi$, and hence $(i, \tilde{i})$ is in the pullback.

iii *The pullback of an acyclic fibration along any other map is again an acyclic fibration.* Let $F : \mathcal{C} \to d$ and $G : \mathcal{E} \to \mathcal{D}$ again exact functors, and let G now an acyclic fibration. In the previous part we have shown that $p_0 : \mathcal{C} \times_{\mathcal{D}} \times \mathcal{E}$ is a fibration. It remains to show that it a weak equivalence too if G is. Let us first show that p_0 reflects weak equivalences. Suppose $p_0(f,g) = f$ is a weak equivalence. Then by exactness, Ff is a weak equivalence, and hence Gg is. By the fact that G reflects weak equivalences since it is an acyclic fibration, we have that g is a weak equivalence and hence (f,g) is a weak equivalence in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$.

By our characterization of acyclic fibrations, it remains to show that a fibration $f : Z \to X$, where X comes from a pair (X,Y) in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$, can be lifted to a fibration $(f,g) : (Z,W) \to (X,Y)$ in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$. Since $f : Z \to X$ is a fibration, so is $Ff : FZ \to FX$. Since $FX = GY$ this is a fibration $FZ \to GY$. Because G is an acyclic fibration we can find some fibration $g : W \to Y$ such that $Gg = Ff$ and in particular $GW = FZ$. This gives our fibration (f,g) in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$, and hence p_0 is an acyclic fibration.

iv *Weak equivalences satisfy 2-out-of-3.* Immediate.

v *Isomorphisms are acyclic fibrations.* Immediate.

vi *For every object X there is a path object PX.* Given a category $\mathcal{C}$, write $P\mathcal{C}$ for the path category $(\mathcal{C}^{\bullet \leftarrow \bullet \rightarrow})_{f,h}$. Its objects are diagrams

$$X_0 \xleftarrow[x_0]{} X_{01} \xrightarrow[x_1]{} X_1$$

where both x_0 and x_1 are acyclic fibrations. The map $r : \mathcal{C} \to P\mathcal{C}$ is given by the constant diagram, and $(s,t) : P\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is given by evaluation on 0 and 1. We now have to show that r is a weak equivalence and (s,t) is a fibration. It is clear that both are exact.

(a) *r is a weak equivalence.* It is clear that weak equivalences are reflected. We show that the approximation property holds. With the notation as in that theorem we have a map $f : Y \to FX$, consisting of $f_0 : Y_0 \to X$, $f_1 : Y_1 \to X$ and $f_{01} : Y_{01} \to X$. Now take $Y' = rY_{01}$ and $w = id$. Let $u = rf_{01}$ and v such that $v_0 = y_0$ and $v_1 = y_1$ and $v_{01} = id$. The resulting diagram in $P\mathcal{C}$ commutes, and both v and w are weak equivalences.

(b) (s, t) *is a fibration.*

i. *Isofibration.* Let (i_0, i_1) isomorphisms as in the following diagram

$$\begin{tikzcd}
& X_{01} \arrow[dl, "x_0"'] \arrow[dr, "x_1"] & \\
X_0 \arrow[d, "i_0"] & & X_1 \arrow[d, "i_1"] \\
Y_0 & & Y_1
\end{tikzcd}$$

As i_0 and i_1 are acyclic fibrations, and compositions of acyclic fibrations are acyclic fibrations, putting $Y_{01} = X_{01}$ and $y_0 = i_0 \circ x_0$ and $y_1 = i_1 \circ x_1$ makes it into a valid isomorphism in $P\mathcal{C}$.

ii. *Lifting for factorizations.* Let $f : X \to Y$ a map in $P\mathcal{C}$, and let $f_0 = p_0 \circ w_0$ and $f_1 = p_1 \circ w_1$. Define P_0 to be the pullback of p_0 along y_0. Obtain a map $\langle f_{01}, w_0 \circ x_0 \rangle : X_{01} \to P_0$. Analogous obtain a map $X_{01} \to P_1$. Now let P the pullback of the maps $P_0 \to Y_{01}$ and $P_1 \to Y_{01}$. Obtain a map $X_{01} \to P$, which we factorize as a weak equivalence followed by a fibration which then induces the wanted factorization. Summarized in the following diagram:

$$\begin{tikzcd}
& & X_{01} \arrow[dll, "x_0"] \arrow[d, "w_0"] \arrow[drr, "x_1"] \arrow[dddl, "{\langle f_{01}, w_0 \circ x_0 \rangle}"] \arrow[dddr, "{\langle f_{01}, w_1 \circ x_1 \rangle}"'] & & \\
X_0 \arrow[dd, "w_0"] & & C_{01} \arrow[d] & & X_1 \arrow[dd, "w_1"] \\
& & P \arrow[dl] \arrow[dr] & & \\
C_0 \arrow[d, "p_0"] & P_0 \arrow[l] \arrow[dr] & & P_1 \arrow[r] \arrow[dl] & C_1 \arrow[d, "p_1"] \\
Y_0 & & Y_{01} \arrow[ll, "y_0"] \arrow[rr, "y_1"] & & Y_1
\end{tikzcd}$$

iii. *Lifting for pseudofactorizations.* Let us have a diagram in $\mathcal{C} \times \mathcal{C}$ in the similar notation as the definition. So, given $i_0 : Z_0 \to X_0$ and

$i_1 : Z_1 \to X_1$ we want to define Y_{01} and a map to X_{01}. We do this very similar to the previous proof, by taking the pullbacks of i_0 and x_0 and of i_1 and x_1, and then take the pullback Y_{01} of those pullbacks. What remains now is to apply the lifting property for factorizations.

iv. *Lifting for sections of acyclic fibrations.* This follows from precisely the same argument as the induction step in the proof that acyclic fibrations in diagram categories have sections.

vii *There is a terminal object and every map $X \to 1$ is a fibration.* Immediate.

□

4 The evaluation functor

As mentioned in the introduction, we are interested in showing that the evaluation functor, mapping a fibrant, homotopical semisimplicial object $X_\bullet$ to its set of vertices X_0, is a weak equivalence. Unfortunately we are not yet able to show that this functor is indeed a weak equivalence, but we can give two lemmas which point us in the right direction.

Lemma 16. *The functor* ev_0 *reflects weak equivalences.*

Proof. This follows by repeatedly applying 2-out-of-6. Given $f_\bullet : X_\bullet \to Y_\bullet$ such that f_0 is weak equivalence, we have that $f_0 \circ d_0 = d_0 \circ f_1$, in which everything except f_1 is known to be a weak equivalence, and hence f_1 is. Inductively f_n is a weak equivalence for every n. □

Lemma 17. *The functor* ev_0 *is essentially surjective.*

Proof. Let X in $\mathcal{C}$. Set $X_0 = X$ and $X_1 = PX$ with $(d_0, d_1) = (s, t)$. Now note that we have cone on $[2] /\!\!/ \Delta_+^{\mathrm{op}} \to \mathcal{C}$ with X as vertex and r to every X_1. Obtain a map $\langle r, r, r\rangle : X \to M_2 X$, and factorize it as

$$X \xrightarrow{w_2} X_2 \xrightarrow{x_2} M_2 X$$

with w_2 a weak equivalence and x_2 a fibration. Now suppose we have obtained X_n in a similar manner, as in a factorization

$$X \xrightarrow{w_n} X_n \xrightarrow{x_n} M_n X$$

We obtain a cone on $[n+1] /\!\!/ \Delta_+^{\mathrm{op}}$ with vertex X and with maps w_n to each X_n. Factorize the resulting map $X \to M_{n+1}X$ to obtain X_{n+1}. Inductively this defines a fibrant homotopical semisimplicial object $X_\bullet$. □

However, the induced map on homotopy categories being full and faithful turned out to be hard. For example when trying show fullness, one has to extend a map $f_0 : \mathrm{ev}_0(X_0) \to \mathrm{ev}_0(Y_0)$ to a morphism of semisimplicial objects $\tilde{f}_\bullet : X_\bullet \to Y_\bullet$ such that f_0 is homotopic to $\tilde{f}_0$. Defining such a map is not easy, since the semisimplicial structure has to be respected, which asks for many homotopy coherence rules to be satisfied. The following lemma shows how one can do the first level of such a construction.

Lemma 18. *We can extend a map $f_0 : \mathrm{ev}_0(X_\bullet) \to \mathrm{ev}_0(Y_\bullet)$ to a pair of maps $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ such that they form the bottom of a morphism of semisimplicial objects.*

Proof. First, observe that $d_i \simeq d_j$ for every two face maps with same domain and codomain. One can see this for the higher degree face maps by using the simplicial identities for $d_i \circ d_i$, being $d_i \circ d_{i+1}$. Because d_i is a weak equivalence it is a homotopy equivalence and hence

$$d_i^{-1} \circ d_i \circ d_i \simeq d_i^{-1} \circ d_i \circ d_{i+1}$$

and hence $d_i \simeq d_{i+1}$. That $d_0, d_1 : X_1 \to X_0$ are homotopic follows because $d_0 \circ d_2 = d_1 \circ d_0$, and these higher d_2 and d_1 were proven to be homotopic.

So, given $f_0 : X_0 \to Y_0$, define $\tilde{f}_1 := d_0^{-1} \circ f_0 \circ d_0$. The following diagram commutes up to homotopy

$$\begin{tikzcd} X_1 \arrow[r, "\widetilde{f_1}"] \arrow[d, two heads, "(d_0, d_1)"] & Y_1 \arrow[d, two heads, "(d_0, d_1)"] \\ X_0 \times X_0 \arrow[r, "f_0 \times f_0"] & Y_0 \times Y_0 \end{tikzcd}$$

and hence by Proposition 2.31 in [14] we can replace $\tilde{f}_1$ by f_1 such that it commutes strictly. □

5 Conclusion and further research

We have shown in these notes that, as one can expect, many of the results proven on fibration categories and tribes can be also proven for path categories. This gives the first steps into a proof that every object internally is an ∞-groupoid. Proving that the evaluation functor is a weak equivalence turned out to be harder than expected. However, there is hope. In [10], Schwede has proven that this evaluation functor is a weak equivalence for cosemisimplicial objects in cofibration categories. Dualizing the appropriate parts, we can use his proof to prove that for path categories this also holds. In fact, there is a high chance that the proof can be simplified, since our path categories have a stronger structure than fibration categories.

Something else worth looking at is whether the approximation property, which we used to characterize the weak equivalences, can be modified to a somewhat simpler statement. In path categories, the homotopy equivalences have a much cleaner description than in an ordinary fibration category; they are precisely the weak equivalences.

It is also interesting to investigate whether the functor ev_0 is a fibration. At a first glimpse, one can prove easily that the functor is an isofibration, and that the lifting property for factorizations is satisfied. Proving this, the proof that ev_0 is a weak equivalence too is made easier by the characterization of acyclic fibrations we gave in section 3.

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