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Path categories and ∞-groupoids

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Abstract

The main object of study in this thesis are path categories. Path categories are categories with classes of fibrations and weak equivalences satisfying a list of axioms. One particular example is that of the syntactic category of a version of type theory called objective type theory. Motivated by the sketch of a proof of homotopy canonicity by Sattler and Kapulkin for homotopy type theory we establish new properties of path categories and its homotopy theory, showing that the ideas of their proof can also be used for objective type theory. Besides this, we give two proofs that every object in a path category has the structure of an internal ∞-groupoid. Although the main motivations of this thesis are of type theoretical nature, this thesis can be read without any knowledge on type theory, as all results are purely categorical.

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1 Introduction

Intuitionistic type theory is a foundational system for mathematics which was introduced in the early '70s by Swedish mathematician Martin-Löf. It is a further developed version of the theory of types, which was introduced by Russell to overcome the paradoxes of set theory. Intuitionistic type theory is the study of *types* and *terms* belonging to a type. One can think of types as sets and terms as objects. However, in contrast to set theory, where an element can appear in different sets, a term belongs to one and one type only. Another difference between set theory and intuitionistic type theory is that all terms and types are explicitly defined or constructed, hence the adjective *intuitionistic*. An example is the type of natural numbers $\mathbb{N}$, which can be characterized as the smallest closure of the term 0 and successor terms $S(n)$ for every n in $\mathbb{N}$. A good introduction to the subject is given in [1].

An important idea in intuitionistic type theory is that *propositions are types*. This means that given a mathematical statement or proposition, there should exist a type whose terms are proofs of that proposition. In fact, we identify a proposition with its collection of proofs. We would like to point out one specific example of such a proposition. Given a type A and terms a, b of type A, there should exist a type

$$\mathrm{Id}_A(a, b)$$

whose terms are *proofs that a and b are equal as terms of A*. We call this type the *identity type*, and the terms a and b *propositionally equal* if the identity type is non-empty. The terms of the identity types are generated by the canonical proof $r(a)$ of type $\mathrm{Id}_A(a, a)$. For explicit syntax of the identity type we refer to [2].

Something wonderful happens if one thinks of a type A as a topological space and terms a, b as points in this space. The identity type $\mathrm{Id}_A(a, b)$ then can be thought of as the *space of paths between a and b*. This gives a type A roughly the structure of a *groupoid*, as every path has an inverse up to some higher structure. Using this interpretation of type theory, Hofmann and Streicher have shown in their illuminating paper [3] that the category of groupoids can serve a *model* or *categorical semantics* of intuitionistic type theory. In other words, we can interpret the formal language of type theory in a category in such a way that categorical operations have a meaningful type theoretic interpretation and vice versa.

The geometrical interpretation of type theory does not end at the groupoid model of Hofmann and Streicher. Given two terms ω, γ of type $\mathrm{Id}_A(a, b)$, one can construct the type

$$\mathrm{Id}_{\mathrm{Id}_A(a,a)}(\omega, \gamma)$$

consisting of *homotopies between paths*. Iterating this yields the idea that *types are ∞-groupoids*, which has been formally shown by Van den Berg and Garner in [4]. In

[5] this concept has been worked out in another way: they show that type theory can be modelled in the category of simplicial sets. We see that there is an evident relation between type theory and algebraic topology, by means of the categorical models of type theory which have a topological nature. This relation has been foundational in the development of *homotopy type theory*, which is a variant of type theory with an extra axiom called *univalence*. Univalence is a formalization of the idea that mathematical structures which can be identified up to some notion of equivalence or isomorphism should be considered as the same structure. A more thorough examination of the philosophical nature of the univalence axiom can be found in Awodey's paper [6].

Another way to interpret type theory in category theory is by making use of the so called *syntactic category*. Instead of modelling the language in a *pre-existing* category, as done in the previous paragraph, we *construct* a category from the type theory. Roughly the objects in the syntactic category of type theory are types, and morphisms between objects are specific choices of terms. We refer to the second chapter in [7] for an explicit definition of this syntactic category. Certain constructions in type theory induce objects satisfying a universal property in the syntactic category. An example is the aforementioned natural numbers type, which for standard intuitionistic type theory is interpreted by the natural numbers object in its syntactic category. Another example is that of the product type, which is interpreted by the usual categorical product.

Modifying the rules of the type theory or language modifies the properties of its corresponding syntactic category. There are many variations of type theory, each with different *computation rules* and *type formers*. In this thesis we are interested in a variant of type theory which is coined *objective type theory* in [8]. Objective type theory is conceptually similar to homotopy type theory, but has a slightly different syntax; in objective type theory every appearance of *judgemental equality* is eliminated and replaced by propositional equality. Two terms a, b in A are judgemental equal if the statement $a = b : A$ is derivable in the deductive system of the type theory. By the nature of objective type theory the constructions in the type theory do not yield objects with standard universal properties in its syntactic category, but objects with *homotopy universal properties*. These structures are not strictly unique but are unique up to a suitable notion of weak equivalence.

Abstracting the syntactic category of objective type theory yields what is called a *path category*, originally introduced by Van den Berg and Moerdijk in [9], and which will be the main object of study in this thesis. The structure of a path category is related to that of a *model category* as introduced in [10], in the sense that there is a class of *fibrations* and *weak equivalences* satisfying a list of properties. However, in contrast to model categories, path categories do not have a weak factorization system, but only something which is weaker: for a commuting square with a fibration on the right and a weak equivalence on the left, there is a diagonal filler which makes the lower triangle commute strictly, but the upper triangle only up to a certain notion of *fibrewise homotopy*. A main example of a path category, besides the syntactic category of objective type theory, is the category of topological spaces with fibrations being the Hurewicz fibrations and the weak equivalences the homotopy equivalences. This reflects again the relation between algebraic topology and type theory.

Categorical semantics of type theory can be used to prove purely type theoretic statements. An example is that of *canonicity*, proven in [11]. Canonicity states that, given a certain type theory with a natural numbers type $\mathcal{N}$, every closed term of the natural numbers type is of the form $S^n(0)$, where S denotes the successor operation.

Unfortunately, type theories which are related to objective type theory, such as homotopy type theory, do not satisfy canonicity. However, it is conjectured that they satisfy what is called *homotopy canonicity*: for every closed term a of natural numbers type $\mathcal{N}$, one can construct a term p of the identity type $\mathrm{Id}_{\mathcal{N}}(a, S^n(0))$ for some n. Proving homotopy canonicity would be interesting for both computational aspects of type theory as well as its philosophical foundations. As type theory is used for proof checking software, and homotopy type theory is as well, a constructive proof of homotopy canonicity would give a way of transforming terms in types of canonical form. Philosophically, homotopy canonicity would show that in some sense the univalence axiom does not violate *constructivity* completely, as mentioned by Voevodsky in [12].

As mentioned, homotopy canonicity is currently unproven for objective type theory. However, a proof that homotopy type theory satisfies homotopy canonicity has been announced by Sattler and Kapulkin in their presentation [13]. The outline of their proof suggests that this proof could very well also work for path categories and objective type theory. Such a proof can be divided in two parts. There is a purely homotopy theoretical side of the proof in which certain properties of the categorical semantics, in our case path categories, are described. On the other hand there should be a soundness and completeness result for the interpretation of the type theory in the categorical semantics. In Section 4.6 we will give an informal outline of the proof.

We can now state the main motivation of this thesis, which is to provide all the homotopy theory needed for the proof of homotopy canonicity. Explicitly this will mostly consist of studying the category **Pth** of path categories and functors between them preserving the path category structure, the *exact functors*. In particular we will study the homotopy theory of **Pth** by endowing it with the structure of a *fibration category*, which itself is a categorical structure similar to that of a path category.

However, the value of the results presented in this thesis is not limited to their role in the proof of homotopy canonicity, but is also of independent interest. Among the main contributions of this thesis are two proofs that objects in a path category carry the structure of an ∞-groupoid. This shows that types in objective type theory have the structure of an ∞-groupoid, which reflects the idea that many of the results on homotopy type theory can be derived for objective type theory as well, as mentioned in [8].

Let us give a summary of the content and main contributions of this thesis:

> **Chapter 2** In Chapter 2 we give an introduction to path categories and review its basic properties. We also describe a class of categories I, the *inverse categories*, for which given a path category $\mathcal{C}$ we can endow the functor category $[I, \mathcal{C}]$ with notions of fibrations and weak equivalences such that $[I, \mathcal{C}]_{\mathrm{f}}$, the full subcategory on the fibrant objects, forms a path category. Moreover we will describe the fibration category of path categories. The main contribution presented in this chapter is

that acyclic fibrations of path categories are surjective on objects and full.

Chapter 3 This chapter is devoted to the higher categorical properties of path categories. We start by a quick review of theory on semisimplicial sets and we introduce the notion of a *frame*. For a path category $\mathcal{C}$ the category of frames $\mathrm{Fr}\,\mathcal{C}$ consists of fibrant semisimplicial objects in $\mathcal{C}$ in which every component is a weak equivalence. In particular, this category $\mathrm{Fr}\,\mathcal{C}$ is a path category. By endowing the category of semisimplicial sets with appropriate notions of fibrations and weak equivalences, we get a path category $\mathbf{ssSet}_{\mathrm{f}}$ of fibrant semisimplicial sets, or semisimplicial Kan complexes. Then, we will consider a pairing of weak equivalences and semisimplicial sets, which we will use to mimic a proof of Schwede presented in [14] of the fact that the canonical evaluation functor $\mathrm{ev}_0 : \mathrm{Fr}\,\mathcal{C} \to \mathcal{C}$ is a weak equivalence of path categories. The main contribution of this chapter consists of a proof that the emphglobal sections functor $\mathrm{Fr}\,\mathcal{C} \to \mathbf{ssSet}_{\mathrm{f}}$, induced by the functor $\mathrm{Hom}_{\mathcal{C}}(1, -) : \mathcal{C} \to \mathbf{Set}$, is an exact functor of path categories. Together with the fact that ev_0 is a weak equivalence this gives the first proof to the statement that objects in path categories are internal ∞-groupoids. We will then review some notions of enriched categories in the context of path categories, starting by enriching the category $\mathrm{Fr}\,\mathcal{C}$ over semisimplicial sets inspired by the works of Kapulkin and Szumilo [15]. We show that in the case of path categories this enrichment is valued in fibrant semisimplicial sets. By work of Den Besten in [16] it can be shown that there is a functor $M : \mathbf{Pth} \to \mathbf{GpdCat}$ which sends every path category to a canonical groupoid enriched category. We will show that for an appropriate fibration category structure on $\mathbf{GpdCat}$ this functor is almost an exact functor. The final contribution of this chapter is a proof that we can compare the aforementioned enrichments by a semisimplicial counterpart of the *fundamental groupoid*.

Chapter 4 In this chapter we study objects with homotopy universal properties in path categories and give an informal summary of the proof of homotopy canonicity, and how this thesis ties in with this proof. The contributions presented in this chapter consists of proofs that the homotopy universal properties are stable under weak equivalences of path categories.

Chapter 5 The "stranger in the midst" of this thesis is Chapter 5, in the sense that it is not directly related to the proof of homotopy canonicity. In this chapter we solve an open problem posed by Lobski in [17]. Using techniques introduced in [18] we show that every object in a path category has the structure of an internal Grothendieck ∞-groupoid in the sense of Maltsiniotis' paper [19]. The latter is a cleaner version of the definition of an internal ∞-groupoid as presented by Batanin in [20], which is used by Van den Berg and Garner in [4]. The advantage of this definition of an ∞-groupoid is that it reflects more of the nature of type theory than the definition we use in Chapter 3, as described in the introduction of [4].

2 Path categories and the fibration category of path categories

This chapter is meant to be an introduction to path categories and the fibration category of path categories. The first four sections are by no means original, and are mostly based on, or cited from existing literature; mainly [9]. In the last two sections we derive certain properties of the fibration category of path categories which are new and prove themselves to be very useful in what is to come in later chapters.

2.1 Path categories

In this section we will give the definition of a path category and some of its basic properties. For a more comprehensive introduction we refer to [9].

Definition 2.1. A *path category* is a category $\mathcal{C}$ with two designated classes of maps: the *fibrations* and the *weak equivalences*. The maps which are both a fibration and a weak equivalences are called *acyclic fibrations*. The following axioms should be satisfied:

(i) Fibrations are closed under composition.

(ii) Pullbacks of fibrations exist and are fibrations again.

(iii) Pullbacks of acyclic fibrations are acyclic fibrations.

(iv) Weak equivalences satisfy 2-out-of-6. That is, if we have three composable maps fgh such that fg and gh are weak equivalences, then all f, g, h and fgh are weak equivalences.

(v) Isomorphisms are acyclic fibrations and every acyclic fibration has a section.

(vi) For every object X there is a *path object* PX. That is, there is a factorization

$$X \xrightarrow{r} PX \xrightarrow{(s,t)} X \times X$$

of the diagonal $\Delta : X \to X \times X$, with r a weak equivalence and (s, t) a fibration.

(vii) $\mathcal{C}$ has a terminal object 1 and every map $X \to 1$ is a fibration.

Remark 2.2. We observe some immediate consequences of the definitions. We first note that 2-out-of-6 implies the more well known 2-out-of-3 statement, which says that for two composable maps f and g, we have that if any two of the maps f, g, fg are weak equivalences, the third is. It is also relevant to note, and implicitly used in axiom (vi), that all products exist; this follows from axioms (ii) and (vii). In particular, the projection maps are fibrations. Moreover we note that by the fact that weak equivalences satisfy 2-out-of-6 it follows that both s and t are weak equivalences and hence acyclic fibrations. We will refer to these maps as *source* and *target* respectively. To the map r we will refer as the *constant path map*.

The axioms of a path category are an extension of the axioms of a category of fibrant objects as introduced by Brown in [21].

Definition 2.3. A *category of fibrant objects*, or a *fibration category*, is a category $\mathcal{C}$ with two designated classes of maps: the fibrations and the weak equivalences, satisfying the following axioms:

(i) Fibrations are closed under composition.

(ii) Pullbacks of fibrations exist and are fibrations again.

(iii) Pullbacks of acyclic fibrations are acyclic fibrations.

(iv) Weak equivalences satisfy 2-out-of-3.

(v) Isomorphisms are acyclic fibrations.

(vi) For every object X there is a path object PX.

(vii) $\mathcal{C}$ has a terminal object 1 and every map $X \to 1$ is a fibration.

It is clear that every path category is a fibration category. The converse does not hold. We will see a counterexample in Example 2.11. Both definitions are related to the definition of a *model category*, as introduced by Quillen in [10], in which there are three designated classes of maps: fibrations, weak equivalences and cofibrations. Given a model category, the full subcategory of fibrant objects, which are the objects whose unique map to the terminal object is a fibration, yields a fibration category. If moreover every object in the model category is cofibrant, this subcategory is a path category. This gives already many examples of path categories, such as the category of topological spaces with Hurewicz fibrations and homotopy equivalences. In this path category, the path objects are exactly as one expects: given a topological space X, the space of continuous functions from the interval I to X endowed with the compact open topology is a path object on X. Another example of a path category arising from a model category is the category of Kan complexes, which are the fibrant objects the category of simplicial sets with the Kan-Quillen model structure.

An example of a path category which does not necessarily arises as the subcategory of a model category, but which will be relevant in this thesis, is the syntactic category for a certain version of type theory. For intuitionistic type theory with function extensionality this is proven by Avigad in [22]. We refer to [23] for a detailed exposition of function extensionality. For the version of type theory we are mostly interested in, which is objective type theory, it is shown by Van den Berg and Den Besten in [8] that its syntactic category is a path category.

Without proof we state a few important properties of path categories. We remark that all lemmas in this section also hold true in fibration categories.

Lemma 2.4. *Every morphism $f : X \to Y$ in a path category factorizes as $f = p_f w_f$, where p_f is a fibration and w_f is a section of an acyclic fibration.*

Proof. Proposition 2.3 in [9]. □

It follows directly that every weak equivalence factorizes as a section of an acyclic fibration followed by an acyclic fibration.

Given a path category $\mathcal{C}$ and an object X in $\mathcal{C}$, the slice category $\mathcal{C}_{/X}$ does not have to carry a path category structure. However, if we take the full subcategory $(\mathcal{C}_{/X})_{\mathrm{f}}$, consisting of the fibrations with codomain X, we obtain a path category. The weak equivalences and the fibrations are defined as the ones reflected by the forgetful functor $U : (\mathcal{C}_{/X})_{\mathrm{f}} \to \mathcal{C}$.

For any morphism $f : X \to Y$ there is an induced functor $f^* : (\mathcal{C}_{/Y})_{\mathrm{f}} \to (\mathcal{C}_{/X})_{\mathrm{f}}$, because pullbacks along fibrations exist. We have the following lemma.

Lemma 2.5. *Let $f : X \to Y$ a morphism in a path category $\mathcal{C}$. The functor $f^* : (\mathcal{C}_{/Y})_f \to (\mathcal{C}_{/X})_f$ preserves fibrations, weak equivalences, pullbacks along fibrations and the terminal object.*

Proof. Lemma 4.1 in [21]. □

We can use this to prove the following lemma:

Lemma 2.6. *Let $f : X \to Y$ a fibration in a path category and let $w : Z \to Y$ a weak equivalence. The pullback of w along f is a weak equivalence.*

Proof. Lemma 4.2 in [21]. □

All end this section with the so called cube lemma for path categories.

Lemma 2.7. *Let $\mathcal{C}$ a path category and let*

$$\begin{tikzcd} B \arrow[r, "p"] \arrow[d, "v_B"'] & A \arrow[d, "v_A"] & C \arrow[l, "f"'] \arrow[d, "v_C"] \\ B' \arrow[r, "p'"'] & A' & C' \arrow[l, "f'"] \end{tikzcd}$$

a pair of commutative diagrams, with p and p' fibrations and v_A, v_B and v_C weak equivalences. Then the induced map

$$B \times_A C \to B' \times'_A C'$$

is a weak equivalence.

Proof. Dualizing Proposition 2.2.12 in [18]. □

2.2 Homotopy and transport

In this section we will discuss notions of homotopy and transport in path categories. Transport is an important concept in type theory. Whereas its type theoretic definition is rather ad hoc and depends heavily on syntax, we can introduce it in the language of path categories in a more conceptual and natural way. Without further ado, we define homotopy in path categories.

Definition 2.8. Let $f, g : X \to Y$ parallel morphisms in a path category $\mathcal{C}$. Then f and g are *homotopic*, denoted by $f \simeq g$, if there is a morphism $H : X \to PY$ such that $sH = f$ and $tH = g$. The map H is called the *homotopy*.

Remark 2.9. It is a matter of computation to show that homotopy is both an equivalence relation and a congruence relation. Being a congruence relation states that for composable f and k and l and g we have that $f \simeq g$ and $k \simeq l$ implies $kf \simeq lg$. Moreover, homotopy is independent of the choice of path object. In particular this means that we can make sense of $\mathrm{Ho}\,\mathcal{C}$, the *homotopy category* of $\mathcal{C}$. The objects of $\mathrm{Ho}\,\mathcal{C}$ are the same as the objects of $\mathcal{C}$, and the morphisms are equivalence classes of morphisms in $\mathcal{C}$ identified up to homotopy.

The morphisms in $\mathcal{C}$ which become isomorphisms in $\mathrm{Ho}\,\mathcal{C}$ are called *homotopy equivalences*. In other words, a map $f : X \to Y$ is a homotopy equivalence if there is some $g : Y \to X$ such that $fg \simeq id_Y$ and $gf \simeq id_X$. We have the following characterization of the homotopy equivalences in $\mathcal{C}$.

Theorem 2.10. *Weak equivalences and homotopy equivalences coincide in a path category $\mathcal{C}$.*

Proof. Theorem 2.16 in [9]. □

Let us now give an example of a fibration category which is not a path category:

Example 2.11. Consider the model structure on topological spaces with fibrations being the Serre fibrations and weak equivalences the weak homotopy equivalences. Every object in this category is fibrant and hence it forms a fibration category. However, not every weak homotopy equivalence is a homotopy equivalence. An example of this can be found in Exercise 10 in Section 4.1 in Hatcher [24].

Before we introduce transport we need the notion of fibrewise homotopy.

Definition 2.12. Let $p : Y \to I$ a fibration in a path category $\mathcal{C}$. We write $P_I Y$ for the path object of $p : Y \to I$ in $(C_{/I})_{\mathrm{f}}$, and call this object the *fibred path object* of Y over I.

In particular the fibred path object of a fibration $p : Y \to I$ is a factorization of the map $\langle id, id \rangle : Y \to Y \times_I Y$.

Definition 2.13. Let $f, g : X \to Y$ parallel morphisms in a path category $\mathcal{C}$, and let $p : Y \to I$ a fibration such that $pf = pg$. Then f and g are *fibrewise homotopic* over I, denoted by $f \simeq_I g$, if there is a *fibrewise homotopy* $H : X \to P_I Y$ such that $\langle s, t \rangle H = \langle f, g \rangle$ as maps $X \to Y \times_I Y$.

Remark 2.14. We observe a few important facts. Ordinary homotopy in a path category is fibrewise homotopy with respect to the unique fibration to the terminal object. If the map fp as in Definition 2.13 is a fibration, then fibrewise homotopy over p is the same as ordinary homotopy in $(C_{/I})_{\mathrm{f}}$.

Lemma 2.15. *Let $f, g : X \to Y$ parallel morphisms in a path category $\mathcal{C}$, and let $p : Y \to I$ an acyclic fibration such that $pf = pg$. Then f and g are fibrewise homotopic over I.*

Proof. Let $Y \times_I Y$ the product of $p : Y \to I$ in $(C_{/I})_{\mathrm{f}}$. By 2-out-of-3 the diagonal $\langle id, id \rangle : Y \to Y \times_I Y$ is a weak equivalence. It follows that $\langle s, t \rangle : P_I Y \to Y \times_I Y$, where $P_I Y$ is the fibred path object over I, is an acyclic fibration. Let u its section. Now $u \circ \langle f, g \rangle : X \to P_I Y$ is the fibrewise homotopy between f and g. □

Fibrewise homotopy lacks the congruence properties mentioned in Remark 2.9. However, the following lemmas will give some machinery which will make it easier to identify fibrewise homotopic maps.

Lemma 2.16. *Let $f, g : X \to Y$ to maps and $p : Y \to I$ a fibration such that $f \simeq_I g$. Let $u : Z \to X$ any map. Then $fu \simeq_I gu$.*

Proof. Let $H : X \to P_I Y$ a fibrewise homotopy between f and g. It is clear that $Hu : Z \to P_I Y$ is a fibrewise homotopy between fu and gu. □

Lemma 2.17. *Let*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[b]{} & Y \end{array}$$

a commutative diagram with q and p fibrations, and let $f, g : Z \to A$ two maps which are fibrewise homotopic over B. Then $af \simeq_Y ag$.

Proof. Corollary 2.12 in [16]. □

Lemma 2.18. *Let*

$$\begin{array}{ccc} B \times_Y X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[b]{} & Y \end{array}$$

a pullback square with p a fibration. To show that two maps $f, g : Z \to B \times_Y X$ are fibrewise homotopic over B, it suffices to show that $\pi_0 f = \pi_0 g$ and $\pi_1 f \simeq_Y \pi_1 g$.

Proof. Remarks 2.1.9 and 2.2.6 in [25]. □

Lemma 2.19. *Let $f, g : X \to Y$ two maps and $p : Y \to I$ a fibration. Suppose $w : Z \to X$ is weak equivalence such that $fw \simeq_I gw$. Then $f \simeq_I g$.*

Proof. Lemma $A.5$ in [7]. $\square$

Let us now give a definition of a transport structure on a fibration:

Definition 2.20. Let $f : X \to Y$ a fibration in a path category $\mathcal{C}$. A *transport structure* on f is a morphism $\Gamma : X \times_Y PY \to X$, where $X \times_Y PY$ is the pullback

$$\begin{tikzcd} X \times_Y PY \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & PY \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

such that $f\Gamma = t\pi_1$ and $\Gamma\langle 1_X, rf\rangle \simeq_Y 1_X$.

The idea of transport is the following. For a path $\alpha : y \to y'$ in Y and a point x in X such that $f(x) = y$, we can transport x along the path α to obtain x' such that $f(x') = y'$. Moreover, the path connecting x and x' should lie entirely in the fibre over y if α is the constant path ry. The following theorem ensures the existence of transport.

Theorem 2.21. *Let $f : X \to Y$ a fibration in a path category $\mathcal{C}$. Then f carries a transport structure which is unique up to fibrewise homotopy over Y.*

Proof. Theorem 2.26 in [9]. $\square$

As mentioned in the introduction, we do not have a weak factorization system in a path category. However, we do have the following theorem on liftings in certain diagrams.

Theorem 2.22. *Let $\mathcal{C}$ a path category and let*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "w"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"'] & Y \end{tikzcd}$$

a commutative diagram with w a weak equivalence and f a fibration. Then there exists a morphism $l : B \to X$ such that $lw \simeq_Y a$ and $fl = b$. The map l is unique up to fibrewise homotopy over Y.

Proof. Theorem 2.38 in [9]. $\square$

An important ingredient in the proof of Theorem 2.22 is the following proposition.

Proposition 2.23. *Let $\mathcal{C}$ a path category and let*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ B \arrow[ur, "l"] \arrow[r, "b"'] & Y \end{tikzcd}$$

a triangle commuting up to homotopy, and let f be a fibration. Then there is a map l' homotopic to l such that $fl' = b$.

Proof. Proposition 2.31 in [9]. $\square$

We can use this proposition to prove the following lemma

Lemma 2.24. *Let $\mathcal{C}$ a path category. Suppose that $(p_0, p_1) : X \to Y \times Y$ is a fibration whose components are both acyclic fibrations. then X is a path object on Y if and only if $p_0 \simeq p_1$.*

Proof. It is clear that X being a path object on Y implies that p_0 and p_1 are homotopic. Let us now prove the other implication. Because p_0 is an acyclic fibration it has a section $u : Y \to X$. Consider the following diagram:

$$\begin{tikzcd} & X \arrow[d, "{(p_0,p_1)}"] \\ Y \arrow[ur, "u"] \arrow[r, "\Delta"'] & Y \times Y. \end{tikzcd}$$

It commutes up to homotopy since p_0 and p_1 are homotopic and hence p_0u and p_1u are. By Proposition 2.23 we obtain a common section of p_0 and p_1 and conclude that X is a path object on Y. $\square$

We end this section with the following characterization of acyclic fibrations in a path category.

Lemma 2.25. *Let $f : X \to Y$ a fibration in a path category $\mathcal{C}$. Then f is an acyclic fibration iff there is a section s of f such that $sf \simeq_Y id$.*

Proof. Proposition 2.33 in [9]. $\square$

2.3 Diagrams in path categories

In this section we will describe a class of indexing categories I for which we can endow (a certain subcategory of) the functor category $[I, \mathcal{C}]$ with a path category structure if $\mathcal{C}$ is a path category. These results have been worked out for type theoretic fibration categories, which are very similar to path categories, in [26]. The proofs for path categories are nearly identical, and are worked out in [27]. We will summarize the results.

Definition 2.26. We call a category I an *inverse category* if the relation "y receives a non-identity arrow from x" is well founded. We write $y \prec x$ for this relation. The *rank* $\rho(x)$ of an object x in I is defined as the supremum

$$\sup_{y \prec x}(\rho(y) + 1).$$

We define the rank of the category I as

$$\sup_{x \in I}(\rho(x) + 1).$$

Given an object x in an inverse diagram, we write $x /\!\!/ I$ for the subcategory of the coslice category x/I in which we exclude the identity.

Definition 2.27. Let I an inverse category and $\mathcal{C}$ any category. Suppose A is a diagram defined on the subcategory $\{y : y \prec x\}$. We write M_xA for the limit, if it exists, of the diagram of $x /\!\!/ I \to \mathcal{C}$. We call M_xA the *matching object* of A at x.

If for some inverse category I a diagram A has been defined on the subcategory $\{y : y \prec x\}$ and the matching object M_xA exists, it suffices to define A_x and a map $A_x \to M_xA$ in order to extend A to $\{y : y \preceq x\}$.

Suppose a natural transformation $f : A \mid_{\{y:y\prec x\}} \to B \mid_{\{y:y\prec x\}}$ is defined on the full subcategory $\{y : y \prec x\}$, and A and B both has matching objects at x. By the universal property there is an induced map $\lim f : M_xA \to M_xB$.

Definition 2.28. Let I an inverse diagram and $\mathcal{C}$ a path category. We call a morphism $f : A \to B$ between two diagrams of I in $\mathcal{C}$ a *(Reedy) fibration* if A and B have all matching objects and the map $A_x \to M_xA \times_{M_xB} B_x$ as in the following diagram:

$$\begin{tikzcd}
A_x \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& M_xA \times_{M_xB} B_x \arrow[r] \arrow[d] & B_x \arrow[d] \\
& M_xA \arrow[r] & M_xB
\end{tikzcd}$$

is a fibration.

For every diagram $A : I \to \mathcal{C}$ there is a unique map to the terminal diagram in $\mathcal{C}$, which is the terminal object in $\mathcal{C}$ in every component. This map is a fibration, or A is fibrant, if all matching objects exists and for every x in I the map $A_x \to M_xA$ is a fibration. We write $[\mathcal{C}, I]_{\mathrm{f}}$ for the subcategory consisting of all the Reedy fibrant diagrams of I in $\mathcal{C}$.

Theorem 2.29. *Let I a small inverse diagram and $\mathcal{C}$ a path category. The category $[\mathcal{C}, I]_{\mathrm{f}}$ with as fibrations the Reedy fibrations and weak equivalences the pointwise weak equivalences is a path category.*

Proof. Combine Theorem 6 and Lemma 7 in [27]. □

Remark 2.30. A property of maps between Reedy fibrant diagrams is the following: if $f : A \to B$ is a weak equivalence respectively fibration then for any object x in I the induced map $\lim f : M_xA \to M_xB$ is a weak equivalence respectively fibration.

One can define an additional class of diagrams in which all morphisms are weak equivalences.

Definition 2.31. Let I any category and $\mathcal{C}$ a path category. We call a diagram A of I in $\mathcal{C}$ *homotopical* if for every map $\alpha : x \to y$ in I the map A_α is weak equivalence.

Proposition 2.32. *Let I a small inverse diagram and $\mathcal{C}$ a path category. The category $[\mathcal{C}, I]_{\mathrm{f,h}}$ of homotopical fibrant diagrams in $\mathcal{C}$ is a path category.*

Proof. Combine Lemma 9 in [27] and Theorem 2.29. □

Example 2.33. In the next section we will describe a fibration category of path categories. Given a path category $\mathcal{C}$ its path object $P\mathcal{C}$ will be the category of fibrant homotopical diagrams of the category

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}$$

in $\mathcal{C}$. Explicitly, these are fibrations $X_{01} \to X_0 \times X_1$ such that both components are weak equivalences (and hence acyclic fibrations).

2.4 The fibration category of path categories

This section is inspired by the work of Kapulkin and Szumilo in [15]. In this paper they compare two frameworks which are similar to path categories: fibration categories and tribes. Tribes were introduced by Joyal in [28]. A tribe is a categorical structure which is a bit stronger than that of a path category, in the sense that every tribe is a path category but not vice versa.

In the paper they compare the frameworks by looking at their homotopy theories. In particular they endow the category of fibration categories and structure preserving functors between them with a fibration category structure, and also endow (a small modification of) the category of tribes with a fibration category structure. In this section we will introduce a category **Pth** of path categories and functors preserving the relevant structure, and describe a fibration category structure in a similar way as done in [15]. For full proofs we refer to [27]. We will also briefly mention how this structure relates to the fibration category of fibration categories. We first give the definition of the "structure preserving functors" between path categories:

Definition 2.34. Let $F : \mathcal{C} \to \mathcal{D}$ a functor between path categories. We call F *exact* if it preserves fibrations, weak equivalences, the terminal object and pullbacks along fibrations.

The notion of an exact functor of fibration categories is identical to that of path categories. It follows from the definition that in both cases exact functors preserve the homotopy relation.

Example 2.35. In Lemma 2.5 we saw that every morphism $f : X \to Y$ in a path category induces an exact functor $f^* : (C_{/Y})_{\mathrm{f}} \to (C_{/X})_{\mathrm{f}}$. It is also clear that every functor from a path category to the terminal category is exact. Another example is the source and target functor $P\mathcal{C} \to \mathcal{C} \times \mathcal{C}$, where $P\mathcal{C}$ is as in Example 2.33. The functor maps a fibration $X_{01} \to X_0 \times X_1$ to the object (X_0, X_1) in $\mathcal{C} \times \mathcal{C}$.

The fibrations between path categories are defined as follows:

Definition 2.36. Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor of path categories. We call F a *fibration of path categories* if it satisfies the following properties:

(i) It is an *isofibration*: for every X in $\mathcal{C}$ and an isomorphism $f' : FX \to Y$ in $\mathcal{D}$, there is an isomorphism $f : X \to Y'$ in $\mathcal{C}$ such that $Ff = f'$.

(ii) It has the *lifting property for factorizations*: Given a morphism $f : X \to Y$ in $\mathcal{C}$, and a factorization $Ff = p'i'$, with p a fibration and i a weak equivalence, there is a factorization $f = pi$ such that $Fp = p'$ and $Fi = i'$.

(iii) It has the *lifting property for sections of acyclic fibrations*: if $f : X \to Y$ is an acyclic fibration in $\mathcal{C}$, and s' is a section Ff, then there is a section s of f such that $Fs = s'$.

It follows easily from the definition that fibrations of path categories can lift acyclic fibrations as in the following lemma:

Lemma 2.37. *Let $F : \mathcal{C} \to \mathcal{D}$ a fibration of path categories. Let X an object in $\mathcal{C}$ and $f' : Y' \to FX$ an acyclic fibration in $\mathcal{D}$. Then there is some $f : Y \to X$ such that $Ff = f'$ and in particular $FY = Y'$.*

Proof. Since f' is an acyclic fibration it has a section $s' : FX \to Y'$. Now the composition $f's'$ is a factorization of the identity on FX. By the lifting property for factorizations we obtain lifts $s : X \to Y$ and $f : Y \to X$ such that $fs = id_X$, the map s is a weak equivalence, and f is a fibration. By 2-out-of-3 it follows that f is an acyclic fibration. □

The notion of a fibration of fibration categories is a little bit different than that of a fibration of path categories. A fibration of fibration categories is an isofibration and has the lifting property for factorizations, but it does not have the lifting property for sections of acyclic fibrations. However, it does satisfy an additional axiom, which is called the lifting property for pseudofactorizations. We show that a fibration of path categories also satisfies this property as a consequence of its axioms.

Lemma 2.38. *Let $F : \mathcal{C} \to \mathcal{D}$ a fibration of path categories. Then F has the* lifting property for pseudofactorizations*, which says that given a morphism $f : X \to Y$ in $\mathcal{C}$ and a diagram*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "s'"'] & FX \arrow[d, "F(f)"] \\ W' \arrow[r, "u'"'] & FY \end{tikzcd}$$

in $\mathcal{D}$, where i' is an acyclic fibration, s' a weak equivalence and u' is a fibration, there exists a diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ W \arrow[r, "u"'] & Y \end{tikzcd}$$

in $\mathcal{C}$, with i an acyclic fibration, s a weak equivalence and u a fibration, such that $Fi' = i$ and $Fs' = s$ and $Fu' = u$.

Proof. By Lemma 2.37 we lift the acyclic fibration $i' : X' \to FX$ to an acyclic fibration $i : Z \to X$. We can now apply lifting for factorizations to u' and s' to obtain lifting for pseudofactorizations. □

We will now define weak equivalences of path categories.

Definition 2.39. Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor of path categories. We call F a *weak equivalence of path categories* if the induced functor

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$$

is an equivalence of categories.

The notion of a weak equivalence of fibration categories is identical to that of a weak equivalence of path categories. Recall that a functor is an equivalence of categories if and only if it is full, faithful and essentially surjective.

Remark 2.40. The functor $\mathrm{Ho}(F)$ being an equivalence is equivalent to saying that F is *homotopy full*, *homotopy faithful* and *homotopy essentially surjective*. This means the following:

Homotopy full For every $f : F(X) \to F(Y)$ there is a map $f' : X \to Y$ such that $F(f') \simeq f$.

Homotopy faithful For every $f, g : X \to Y$ we have that $F(f) \simeq F(g)$ if and only if $f \simeq g$.

Homotopy essentially surjective For every object X in $\mathcal{D}$ there is some X' in $\mathcal{C}$ and a weak equivalence $w : F(X') \to X$.

We have the following characterization of weak equivalences due to Cisinski:

Theorem 2.41. *Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor of path categories. Then F is a weak equivalence if and only if it reflects weak equivalences, and for every $f : Y \to FX$ in $\mathcal{D}$, we can find a map $u : X' \to X$ in $\mathcal{C}$, and weak equivalences $v : Y' \to Y$ and $w : Y' \to FX'$, such that the following diagram commutes:*

$$\begin{tikzcd} Y' \arrow[r, "w"] \arrow[d, "v"'] & FX' \arrow[d, "F(u)"] \\ Y \arrow[r, "f"'] & FX. \end{tikzcd}$$

This last property is also called the approximation property*.*

Proof. Theorem 3.12 in [29]. □

The following lemma, due to Szumilo in [30] for cofibration categories, gives a characterization of acyclic fibrations between path categories.

Lemma 2.42. *An exact functor $F : \mathcal{C} \to \mathcal{D}$ between path categories is an acyclic fibration iff the functor is a fibration, reflects weak equivalences and moreover it satisfies the following property: given a fibration $f : Y \to FX$ in $\mathcal{D}$, there is a fibration $f' : Y' \to X$ in $\mathcal{C}$ such that $Ff' = f$.*

Proof. Lemma 14 in [27]. □

Theorem 2.43. *The category* **Pth** *of path categories and exact functors forms a fibration category, with the fibrations and weak equivalences as defined above.*

Proof. Theorem 15 in [27]. □

The same holds true for category **Fib** of fibration categories and exact functors, with fibrations and weak equivalences as described throughout the section. As we will be studying exact functors in **Fib** with domain **Pth**, it is useful to mention the structure of pullbacks along fibrations in **Pth**. If $F : \mathcal{C} \to \mathcal{D}$ is an exact functor of path categories, and $G : \mathcal{E} \to \mathcal{D}$ is a fibration of path categories, then the underlying category of the pullback $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ is just the ordinary pullback of F and G as functors in **Cat**. The fibrations and the weak equivalences in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ are defined pointwise.

Remark 2.44. As every path category is a fibration category, and every exact functor of path categories is an exact functor of fibration categories, there is an inclusion **Pth** $\to$ **Fib**. It is immediate that this inclusion is an exact functor fibration categories. The functor moreover reflects weak equivalences.

2.5 Exact functors and slice categories

In this section we will show that both fibrations and weak equivalences are stable under slicing. This will be useful for understanding the way fibrewise homotopies behave under these functors.

Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor and let X some object in $\mathcal{C}$. There is an induced functor $F_{/X} : (C_{/X})_{\mathrm{f}} \to (\mathcal{D}_{/F(X)})_{\mathrm{f}}$ which takes an object $p : Y \to X$ in $(C_{/X})_{\mathrm{f}}$ to $F(p) : F(Y) \to F(X)$, which indeed is an object in $(\mathcal{D}_{/F(X)})_{\mathrm{f}}$ as F preserves fibrations.

Lemma 2.45. *Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor of path categories. For any object X in $\mathcal{C}$, the functor $F_{/X} : (C_{/X})_f \to (\mathcal{D}_{/F(X)})_f$ is exact.*

Proof. Immediate. □

Lemma 2.46. *Let $F : \mathcal{C} \to \mathcal{D}$ a fibration of path categories. For any object X in $\mathcal{C}$, the functor $F_{/X} : (C_{/X})_f \to (\mathcal{D}_{/F(X)})_f$ is a fibration.*

Proof. We will prove that $F_{/X}$ is an isofibration if F is, and omit the other proofs. Consider an isomorphism i' over $F(X)$, which is a diagram of the following form:

$$\begin{tikzcd} F(Y) \arrow[rr, "i'"] \arrow[dr, "F(y)"'] & & Z' \arrow[dl, "z'"] \\ & F(X). & \end{tikzcd}$$

Because F is an isofibration we lift i' to $i : Y \to Z$. The diagram

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "y"'] & & Z \arrow[dl, "yi^{-1}"] \\ & X & \end{tikzcd}$$

gives a lift of i as a morphism over $F(X)$. □

Lemma 2.47. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration of path categories. For any object X in $\mathcal{C}$, the functor $F_{/X} : (\mathcal{C}_{/X})_f \to (\mathcal{D}_{/F(X)})_f$ is an acyclic fibration.*

Proof. Because $F_{/X}$ is a fibration by Lemma 2.45, it remains to show that weak equivalences are reflected and every fibration can be lifted as in Lemma 2.42. Reflection of weak equivalences is immediate because F is an acyclic fibration. Now let $f : Z \to F(Y)$ a fibration over $F(X)$ as in

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "z"'] & & F(Y) \arrow[dl, "F(y)"] \\ & F(X). & \end{tikzcd}$$

It is clear that lifting f to some $f' : Z \to Y$ also lifts over X because yf' is a fibration such that $F(yf') = F(y)F(f') = z$. □

Corollary 2.48. *Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence of path categories. For any object X in $\mathcal{C}$, the functor $F_{/X} : (\mathcal{C}_{/X})_f \to (\mathcal{D}_{/F(X)})_f$ is a weak equivalence.*

Proof. First we remark that Lemma 2.4 also holds true in fibration categories. Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence. We factorize F as PW, where P an acyclic fibration and W a section of an acyclic fibration. The functor $P_{/W(X)}$ is an acyclic fibration by 2.47. The functor $W_{/X}$ is a weak equivalence because it is the section of some acyclic fibration G which induces an acyclic fibration $G_{/W(X)}$ such that $G_{/W(X)}W_{/X}$ is the identity and hence by 2-out-of-3 the functor $W_{/X}$ is a weak equivalence and hence $F_{/X}$ is because $F_{/X} = P_{/W(X)}W_{/X}$. □

We will see arguments like this many times in this thesis; to prove certain properties for weak equivalences of path categories it is sufficient to show it for acyclic fibrations.

This is related to something which is called *Brown's Lemma*, although it is a bit different. Brown's Lemma states the following. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor between a fibration category $\mathcal{C}$ and a *category with weak equivalences* $\mathcal{D}$, i.e. a category with a class of weak equivalences satisfying 2-out-of-3 and containing all isomorphisms. Then to show that F preserves weak equivalences, it is enough to show that F maps acyclic fibrations to weak equivalences.

Lemma 2.49. *Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence. Then F is also fibrewise homotopy faithful in the following sense. If $f, g : X \to Y$ are any two maps in $\mathcal{C}$, and $p : Y \to I$ a fibration in $\mathcal{C}$ such that $pf = pg$ and $F(f) \simeq_{F(i)} F(g)$ in $\mathcal{D}$, then $f \simeq_I g$.*

Proof. We have a fibrewise homotopy $H : F(X) \to F(P_I Y)$ such that

$$\begin{tikzcd} & F(P_I Y) \arrow[d, "F\langle s,t\rangle"] \\ F(X) \arrow[ur, "H"] \arrow[r, "\langle F(f),F(g)\rangle"'] & F(Y \times_I Y) \end{tikzcd}$$

commutes. By homotopy fullness there is some H' such that $F(H') \simeq H$. Hence the diagram

$$\begin{tikzcd} & P_I Y \arrow[d, "\langle s,t\rangle"] \\ X \arrow[ur, "H'"] \arrow[r, "\langle f,g\rangle"'] & Y \times_I Y \end{tikzcd}$$

commutes up to homotopy by homotopy faithfulness. We can apply Lemma 2.23 to obtain a map $\widetilde{H}$ which makes the triangle commute. This gives us the desired fibrewise homotopy. $\square$

2.6 2-fibrations

In this section we introduce the notion of a 2-fibration, which is a generalization of the notion of an isofibration between categories to the setting of path categories. We will show that every fibration is in fact a 2-fibration. The main result of this section is Corollary 2.55, in which we show that acyclic fibrations are full and surjective on objects. The motivation for the precise definition of a 2-fibration will become clear in Section 3.7.

Definition 2.50. Let $F : \mathcal{C} \to \mathcal{D}$ an exact functor of path categories. We call F a 2-*fibration* if the following two axioms are satisfied:

(i) For every weak equivalence $w : X \to FY$ in $\mathcal{D}$ there is a weak equivalence $w' : X' \to Y$ such that $Fw' = w$.

(ii) For every homotopy $H : F(X) \to PF(Y)$ between two maps $F(f)$ and g in $\mathcal{D}$, there is a homotopy $H' : X \to PY$ for some path object PY of Y, such that $F(PY) = P(FY)$ and $FH' \simeq_{F(Y)\times F(Y)} H$.

Lemma 2.51. *Suppose an exact functor $F : \mathcal{C} \to \mathcal{D}$ satisfies property (ii) of Definition 2.50. Then property (i) is equivalent to saying that every weak equivalence $FX \to Y$ can be lifted to a weak equivalence $X \to Y'$.*

Proof. We prove that (i) and (ii) imply the the alternative version of (i). Let $w : FX \to Y$ a weak equivalence. We lift $w^{-1} : Y \to FX$ to a map $\widetilde{w^{-1}} : Y' \to X$ in $\mathcal{C}$. Let $\widetilde{w}$ its inverse in $\mathrm{Ho}(\mathcal{C})$. Clearly $F(\widetilde{w}) \simeq w$, and we can lift the homotopy between them to obtain w' such that $F(w') = w$. The other implication has a similar proof. $\square$

Lemma 2.52. *Let $F : \mathcal{C} \to \mathcal{D}$ a fibration. For any object A the functor $F_{/A}$ satisfies the alternative version of property (i) as in Lemma 2.51.*

Proof. Consider a weak equivalence

$$\begin{tikzcd} F(X) \arrow[rr, "w"] \arrow[dr, "F(x)"'] & & Y \arrow[dl, "y"] \\ & F(A) & \end{tikzcd}$$

in $(\mathcal{D}_{/F(A)})_{\mathrm{f}}$. The lifting property for factorizations of F applied to $F(x) = yw$ now yields the weak equivalence in $(\mathcal{C}_{/A})_{\mathrm{f}}$. $\square$

By taking $A = 1$ we have that every fibration satisfies this alternative property as in Lemma 2.51. In fact, every fibration is a 2-fibration.

Theorem 2.53. *Every fibration is a 2-fibration.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ a fibration. We show that property (ii) of Definition 2.50 is satisfied, as this is enough to show F is a 2-fibration by Lemma 2.51 and Lemma 2.52. Let

$$\begin{tikzcd} & FPY \arrow[d, "F(s,t)"] \\ FX \arrow[ur, "H"] \arrow[r, "(Ff,g)"'] & FY \times FY \end{tikzcd}$$

a homotopy in $\mathcal{D}$. We have the pullback

$$\begin{tikzcd} X \times_Y PY \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & PY \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in $\mathcal{C}$, which by F is mapped to a pullback $F(X \times_Y PY)$ in $\mathcal{D}$. We have a map $\langle id, H\rangle : FX \to F(X \times_Y PY)$, which is a section of the acyclic fibration $F(\pi_1)$. By the lifting property for sections of acyclic fibrations, we obtain a map $l : X \to X \times_Y PY$ such that $F(l) = \langle id, H\rangle$. We now claim that $\pi_2 l : X \to PY$ is the homotopy we are looking for. First note that

$$F(\pi_2 l) = F(\pi_2)\langle id, H\rangle = H.$$

Secondly note that

$$s\pi_2 l = f\pi_1 l = f.$$

Lastly note that

$$F(t\pi_2 l) = F(t)H = g$$

and hence F is a 2-fibration. □

Remark 2.54. The proof of Theorem 2.53 shows that fibrations satisfy a homotopy lifting property which is even stronger than axiom (ii) in the definition of a 2-fibration: given a homotopy $H : F(X) \to PF(Y)$ between maps $F(f)$ and g in $\mathcal{D}$, there is a path object PY and a homotopy $H' : X \to PY$ between f and some g' such that $F(PY) = PF(Y)$ and $F(H') = H$ and $F(g') = g$.

Corollary 2.55. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration between path categories. Then F is surjective on objects and full.*

Proof. Surjective on objects follows by applying Lemma 2.42 to the unique fibrations to the terminal object. Now let $f : F(X) \to F(Y)$ in $\mathcal{D}$. We use homotopy fullness to obtain $\widetilde{f} : X \to Y$ such that $F(\widetilde{f}) \simeq f$. Now apply the second property of 2-fibrations to obtain $f' : X \to Y$ such that $F(f') = f$. □

Moreover, acyclic fibrations have some extra lifting properties.

Lemma 2.56. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let $p : X \to Y$ a fibration in $\mathcal{C}$ and $f : Z \to Y$ any map. Suppose there is a map $l : F(Z) \to F(Y)$ such that $F(p)l = F(f)$. Then there is a map $l' : Z \to X$ such that $pl = f$ and $F(l') = l$.*

Proof. Consider the pullback

$$\begin{tikzcd}
Z \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "p"] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}$$

which is preserved by the map F. There is an induced map $\langle id, l \rangle : F(Z) \to F(Z \times_Y X)$. By surjectivity of $F_{/Z}$ there is a map $u : Z \to Z \times_Y X$ which is mapped to $\langle id, l \rangle$. Because we considered the functor to be over Z we have $\pi_1 u = id$. It is also clear that $\pi_2 u$ is mapped to l by functoriality and exactness of F. Now at last we note that

$$p\pi_2 u = f\pi_1 u = f$$

and hence $\pi_2 u$ is indeed as stated in the lemma. □

In particular, acyclic fibrations also have a lifting property for sections of fibrations.

Corollary 2.57. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let $p : X \to Y$ a fibration in $\mathcal{C}$ and let s a section of the fibration $F(p)$. Then there is a section s' of p such that $Fs' = s$.*

Proof. Apply Lemma 2.56 to

$$\begin{tikzcd} & F(X) \arrow[d, "F(p)"] \\ F(Y) \arrow[ur, "s"] \arrow[r, "id"'] & F(Y). \end{tikzcd}$$

□

The diagonal fillers in a commutative diagram with a weak equivalence on the left and a fibration on the right can also be lifted by acyclic fibrations.

Corollary 2.58. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "w"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

a commuting diagram in $\mathcal{C}$ with w weak equivalence and p a fibration. Let $l : F(B) \to F(X)$ a lift such that the lower right triangle commutes and the upper left commutes up to fibrewise homotopy. Then there is a map $l' : B \to X$ such that it does the same for the diagram in $\mathcal{C}$.

Proof. Apply Lemma 2.56 to the lift in question. The result follows by applying Lemma 2.49. □

3 Frames and enrichments

This chapter is devoted to the study of higher structures of path categories. Most of these concepts will play a role in the proof of homotopy canonicity. The main definition of this chapter is that of a *frame* in a path category, which is a semisimplicial object in $\mathcal{C}$ with some extra structure. Frames capture the higher dimensional properties of the objects in a path category in a similar way as the singular complex captures the structure of a topological space.

3.1 Semisimplicial sets and frames

We start by reviewing the notion of a semisimplicial set or object. We refer to [14], [31] or [32] for a more thorough exposition. We will also rely on the results presented in [33]. We assume the reader to be familiar with simplicial sets and the Kan-Quillen model structure.

Definition 3.1. The category Δ_+ is the category of non-empty finite ordinals of the form $[n] = \{0, \ldots, n\}$, and injective monotone functions.

Note that there is a canonical inclusion of Δ_+ into Δ, the category of finite ordinals and all monotone functions.

The presheaf category on Δ_+ is called the category of *semisimplicial sets* and is denoted by **ssSet**. We write $\Delta_+[n]$ for the representable presheaf $\mathrm{Hom}_{\Delta_+}(-, [n])$. A semisimplicial set K is called *finite* if the disjoint union of all simplices is finite. We write **ssSetFin** for the category of finite semisimplicial sets. For an arbitrary category $\mathcal{C}$, we call the functor category $[\Delta_+^{\mathrm{op}}, \mathcal{C}]$ the category of *semisimplicial objects* in $\mathcal{C}$. Given such a semisimplicial object $X_\bullet : \Delta_+^{\mathrm{op}} \to \mathcal{C}$, the face maps satisfy the *semisimplicial identities*

$$d_i d_j = d_{j-1} d_i \quad \text{if } i < j.$$

It is clear that the category Δ_+ is an inverse category and hence we can apply Proposition 2.32 to get the following definition, due to Szumiło in [34]:

Definition 3.2. Let $\mathcal{C}$ a path category. We write $\mathrm{Fr}\,\mathcal{C}$ for the homotopical fibrant diagrams of Δ_+^{op} in $\mathcal{C}$. We call this category the *category of frames in $\mathcal{C}$*.

Lemma 3.3. *Let $X_\bullet$ a frame in a path category $\mathcal{C}$. All parallel face maps in $X_\bullet$ are homotopic.*

Proof. Let $n \geq 2$ and let d_i and d_{i+1} be face maps from X_n to X_{n-1}. By the semisimplicial identities we have that

$$d_i d_{i+1} = d_i d_i$$

and hence $d_i \simeq d_{i+1}$ because $d_i : X_{n-1} \to X_{n-2}$ is a weak equivalence. It remains to show the lemma holds for $d_0, d_1 : X_1 \to X_0$. However, since $d_0 d_2 = d_1 d_0$ and $d_0 \simeq d_2$ as maps $X_2 \to X_1$, it follows immediately that d_0 and d_1 as maps $X_1 \to X_0$ are homotopic. $\square$

There is a functor $\mathrm{ev}_0 : \mathrm{Fr}\,\mathcal{C} \to \mathcal{C}$ mapping a frame $X_\bullet$ to X_0. We will show in Section 3.4 that this functor is an acyclic fibration of path categories.

As in the category of simplicial sets, there are a few important examples of semisimplicial sets:

Definition 3.4. Let n a natural number. We define the *boundary* $\partial\Delta_+[n]$ as the semisimplicial subset of $\Delta_+[n]$ obtained by removing the unique n-simplex of $\Delta_+[n]$. So we have

$$(\partial\Delta_+[n])_m = \begin{cases} (\Delta_+[n])_m & \text{if } m \neq n \\ \emptyset & \text{if } m = n. \end{cases}$$

Now let $0 \leq i \leq n$. We define the *k-horn* $\Lambda^k_+[n]$ as the semisimplicial subset of $\partial\Delta_+[n]$ obtained by omitting the k-th face, or as the semisimplicial subset of $\Delta_+[n]$ consisting of all the faces of the unique n-simplex except the i-th one. So we have

$$(\Delta^k_+[n])_m = \begin{cases} (\Delta_+[n])_m & \text{if } m \notin \{n-1, n\} \\ (\partial\Delta_+[n] \setminus \Delta_+[n-1])_m & \text{if } m = n-1 \\ \emptyset & \text{if } m = n \end{cases}$$

where $\Delta_+[n-1]$ denotes the k-th face.

Semisimplicial sets form a symmetric monoidal category. The tensor product of the monoidal structure is given by the geometric product $\otimes$. The geometric product was originally introduced in [32] to describe the non-degenerate simplicial of the product of two simplicial sets. We will give a brief summary of a more categorical approach of the geometric product, which is given in [31]. Write $\mathbf{Pos}_+$ for the category of posets and injective functions. There is a functor $N_+ : \mathbf{Pos} \to \mathbf{ssSet}$ which maps a poset P to the semisimplicial set $N_+(P)$ whose n-simplices are elements of the set $\mathrm{Hom}_{\mathbf{Pos}_+}([n], P)$. For every $[n]$ we obtain a functor $\Delta_+[n] \otimes - : \mathbf{ssSet} \to \mathbf{ssSet}$ as the left Kan extension of the functor $N_+([n] \times -) : \Delta_+ \to \mathbf{ssSet}$. To obtain $- \otimes X$ for an arbitrary semisimplicial set K we take the left Kan extension of the functor which maps $[n]$ to $\Delta_+[n] \otimes K$. The unit of the monoidal structure is given by the representable semisimplicial set $\Delta_+[0]$.

Given a semisimplicial set K the functor $- \otimes K$ has a right adjoint $[K, -]$ which is the *internal hom* of the monoidal structure. In particular this means that given a triple of semisimplicial sets K, L and M there is an isomorphism

$$\mathrm{Hom}_{\mathbf{ssSet}}(K, [L, M]) \to \mathrm{Hom}_{\mathbf{ssSet}}(K \otimes L, M),$$

which is natural in all three variables. It follows that for two semisimplicial sets L and M the 0-simplices of the internal hom $[L, M]$ are given by elements of the set $\mathrm{Hom}_{\mathbf{ssSet}}(L, M)$.

3.2 A path category of semisimplicial sets

In this section we will describe a path category of semisimplicial sets, based on the works of [31] and [33]. We will then compare the structure with the path category structure on simplicial sets inherited from the Quillen model structure, by means of the canonical functors between simplicial sets and semisimplicial sets induced by the inclusion $\Delta_+ \to \Delta$.

Fibrations between semisimplicial sets will be the semisimplicial analogues of Kan fibrations.

Definition 3.5. Let $f : K \to L$ a morphism of semisimplicial sets. We call f a *semisimplicial Kan fibration* if every diagram of the form

$$\begin{tikzcd} \Lambda^k_+[n] \arrow[r, "\alpha"] \arrow[d] & K \arrow[d, "f"] \\ \Delta_+[n] \arrow[r, "\beta"'] & L \end{tikzcd}$$

has a lift $\Delta_+[n] \to K$ such that both triangles commute strictly. We call f a *semisimplicial trivial Kan fibration* if every diagram of the form

$$\begin{tikzcd} \partial\Delta_+[n] \arrow[r, "\alpha"] \arrow[d] & K \arrow[d, "f"] \\ \Delta_+[n] \arrow[r, "\beta"'] & L \end{tikzcd}$$

has a lift $\Delta_+[n] \to K$ such that both triangles commute strictly.

The category **ssSet** has a terminal object 1 which has precisely one n-simplex for every n. We call a semisimplicial set K a *semisimplicial Kan complex* or *fibrant semisimplicial set* if the unique map $K \to 1$ is a Kan fibration. In particular this means that every horn inclusion $\Lambda^k_+[n] \to K$ can be lifted to an n-simplex $\Delta_+[n] \to K$.

We can define homotopies between morphisms of semisimplicial sets using the geometric product:

Definition 3.6. Two morphisms $\omega, \gamma : K \to L$ of semisimplicial sets are *semisimplicially homotopic* or just *homotopic* if there is a *homotopy* $H : K \otimes \Delta_+[1] \to L$ such that the inclusions of K in $K \otimes \Delta_+[1]$ composed with H yield ω and γ.

The weak equivalences between fibrant semisimplicial sets will now be the homotopy equivalences with respect to the notion of homotopy defined above.

Definition 3.7. Let $f : K \to L$ a morphism of fibrant semisimplicial sets. We call f a *weak equivalence* if there is a map $g : L \to K$ such that fg and gf are semisimplicially homotopic to the identities.

If one wants to extend the definition of a weak equivalence to general semisimplicial sets one has to do a bit more work. This is worked out explicitly in Section 3.3.3 in [31].

Lemma 3.8. *Let $f : K \to L$ a fibration between fibrant semisimplicial sets. Then f is a trivial fibration if and only if it is a weak equivalence.*

Proof. Corollary 3.17 in [33]. □

Theorem 3.9. *The category* $\mathbf{ssSet}_{\mathrm{f}}$ *of fibrant semisimplicial sets forms a fibration category.*

Proof. Theorem 3.18 in [33]. □

Lemma 3.10. *The trivial fibrations fit into a weak factorization system (cofibrations, trivial fibrations), where the cofibrations are the monomorphisms.*

Proof. Section 3.2 in [31]. □

Lemma 3.11. *The category* $\mathbf{ssSet}_{\mathrm{f}}$ *of fibrant semisimplicial sets forms a path category.*

Proof. By Theorem 3.9 the category is a fibration category. By Corollary 3.22 in [33] the weak equivalences satisfy 2-out-of-6. Because the unique map $\emptyset \to K$ is a monomorphism and hence a cofibration, all acyclic fibrations have sections by the weak factorization system in Lemma 3.10. □

By classical theory on presheaf categories the inclusion $\Delta_+ \to \Delta$ induces a forgetful functor $U : \mathbf{sSet} \to \mathbf{ssSet}$ with two adjoints $L, R : \mathbf{ssSet} \to \mathbf{sSet}$. The functors U and R preserve the fibrant objects in both categories, and in particular the path category structure, as in the following theorem.

Theorem 3.12. *The functors* $U : \mathbf{sSet}_{\mathrm{f}} \to \mathbf{ssSet}_{\mathrm{f}}$ *and* $R : \mathbf{ssSet}_{\mathrm{f}} \to \mathbf{sSet}_{\mathrm{f}}$ *are weak equivalences of path categories.*

Proof. Theorem 3.73 in [33] proves that the functors are weak equivalences of fibration categories. However, by Remark 2.44 they are also weak equivalences of path categories. □

3.3 A pairing of semisimplicial sets and frames

In this section we will describe a pairing of semisimplicial sets and frames as introduced in [14] by Schwede for cofibration categories. We will use this pairing in the next section to show that the evaluation functor $\mathrm{ev}_0 : \mathrm{Fr}\,\mathcal{C} \to \mathcal{C}$ is a weak equivalence of path categories.

First, we note that for any semisimplicial object $X_\bullet$ in $\mathcal{C}$ we obtain a functor

$$\mathcal{C}(-, X_\bullet) : \mathcal{C} \to \mathbf{ssSet}$$

which maps an object Z to the semisimplicial set which in degree n is defined by $\mathcal{C}(Z, X_n)$. Now fix some semisimplicial set K and some semisimplicial object $X_\bullet$. We obtain a presheaf on $\mathcal{C}$ which maps an object Z in $\mathcal{C}$ to

$$\mathrm{Hom}_{\mathbf{ssSet}}(K, \mathcal{C}(Z, X_\bullet)).$$

If the presheaf is representable, we write $K \pitchfork X_\bullet$ for its representable object. Alternatively $K \pitchfork X_\bullet$ can be defined as the limit of the following diagram:

$$\int K \to \Delta_+^{\mathrm{op}} \to \mathcal{C}.$$

Recall that $\int K$ is the category of elements, with objects pairs (n, x) such that $x \in K_n$. A morphism $(n, x) \to (m, y)$ is a morphism $d_i : K_n \to K_m$ such that $d_i(x) = m$.

Lemma 3.13. *The two definitions of $K \pitchfork X_\bullet$ are equivalent.*

Proof. Suppose that the functor $\mathrm{Hom}_{\mathbf{ssSet}}(K, \mathcal{C}(-, X_\bullet))$ is represented by $K \pitchfork X_\bullet$. Consider the morphism $\eta : K \to \mathcal{C}(K \pitchfork X_\bullet, X_\bullet)$ corresponding to the identity in $\mathcal{C}(K \pitchfork X_\bullet, K \pitchfork X_\bullet)$. Given an n-simplex x we obtain some $\eta_n(x) : K \pitchfork X_\bullet \to X_n$. Now for a morphism $d_i : (n, x) \to (m, y)$ in $\int K$ we precisely have that $\eta_m \circ d_i = d_i \circ \eta_n$, this makes that $K \pitchfork X_\bullet$ is a cone on the correct diagram. If T is some other limiting cone, then we obtain an element of $\mathrm{Hom}_{\mathbf{ssSet}}(K, \mathcal{C}(T, X_\bullet))$ which then precisely corresponds to a unique element in $\mathrm{Hom}_{\mathcal{C}}(T, K \pitchfork X_\bullet)$. The other implication has a similar proof. $\square$

Example 3.14. Let $X_\bullet$ a semisimplicial object in $\mathcal{C}$. We have the following important examples:

(i) $\Delta_+[n] \pitchfork X_\bullet = X_n$,

(ii) $\partial\Delta_+[n] \pitchfork X_\bullet = M_n X_\bullet$.

Recall that $M_n X_\bullet$ is the matching object of the diagram $X_\bullet : \Delta_+^{\mathrm{op}} \to \mathcal{C}$ at n. Note that (ii) does not necessarily exist for arbitrary $X_\bullet$.

This means a semisimplicial object $X_\bullet$ is fibrant if for every n the pairing $\partial\Delta_+[n] \pitchfork X_\bullet$ exists and the map $\Delta_+[n] \pitchfork X_\bullet \to \partial\Delta_+[n] \pitchfork X_\bullet$, which is induced by the inclusion $\partial\Delta_+[n] \to \Delta_+[n]$, is a fibration.

Lemma 3.15. *The pairing $K \pitchfork X_\bullet$ exists for every fibrant $X_\bullet$ and finite K. Moreover, (acyclic) fibrations between two fibrant semisimplicial objects $X_\bullet$ and $Y\bullet$ are mapped to (acyclic) fibrations in $\mathcal{C}$*

Proof. Proposition 3.3 in [14]. $\square$

By fixing a finite semisimplicial set K or fibrant $X_\bullet$ we obtain two functors, one contravariant and one covariant.

Lemma 3.16. *For fibrant $X_\bullet : \Delta_+^{\mathrm{op}} \to \mathcal{C}$, the functor $- \pitchfork X_\bullet : \mathbf{ssSet} \to \mathcal{C}$ maps colimits to limits. For a finite semisimplicial set K the functor $K \pitchfork - : [\Delta_+^{\mathrm{op}}, \mathcal{C}]_{\mathrm{f}} \to \mathcal{C}$ preserves limits.*

Proof. Proposition 3.4 in [14]. $\square$

Definition 3.17. An inclusion $K \subset L$ of semisimplicial sets is called an *elementary expansion of dimension* n if L is obtained from K by adding one n-simplex and one $n-1$ simplex.

Example 3.18. Archetypical elementary expansions are the horn inclusions $\Lambda^i_+[n] \to \Delta_+[n]$. In fact, every elementary expansion arises as the pushout of an horn inclusion as

Lemma 3.19. *Let $K \to L$ an elementary expansion and $X_\bullet$ some fibrant semisimplicial object in $\mathcal{C}$. The induced map $L \pitchfork X_\bullet \to K \pitchfork X_\bullet$ is an acyclic fibration.*

Proof. Proposition 3.7 in [14]. ☐

Remark 3.20. It is relevant to note that for any frame $X_\bullet$ we have the following pullback square in $\mathcal{C}$:

$$\begin{tikzcd} M_n X_\bullet \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \Lambda^k_+[n] \pitchfork X_\bullet \arrow[r] & M_{n-1} X_\bullet. \end{tikzcd}$$

Because the diagram

$$\begin{tikzcd} \partial\Delta_+[n-1] \arrow[r] \arrow[d] & \Lambda^k_+[n] \arrow[d] \\ \Delta_+[n-1] \arrow[r] & \partial\Delta_+[n] \end{tikzcd}$$

is a pushout in **ssSet**.

3.4 The evaluation functor

In this section we will give a proof that the evaluation functor $\mathrm{ev}_0 : \mathrm{Fr}\,\mathcal{C} \to \mathcal{C}$, mapping a frame $X_\bullet$ to X_0, is an acyclic fibration of path categories. The proof that ev_0 is a weak equivalence, which is Theorem 3.23, and its preliminary lemma which Lemma 3.22, are due to Schwede and can be found in [14]. Let us start by introducing the semisimplicial counterpart of a cone.

Definition 3.21. Let K be a semisimplicial set. We define CK to be the *cone* on K. Its simplices are defined as

$$CK_0 := K_0 \amalg \{*\}$$

and

$$CK_n := K_n \amalg \{\sigma x \mid x \in K_{n-1}\},$$

where the σx are formal elements. The face maps are such that the inclusion $K \subset CK$ is a morphism of semisimplicial objects. On the objects of the form σx in CK_n, they are defined as follows:

$$d_i(\sigma x) = \begin{cases} \sigma d_i(x) & i \neq n \\ x & i = n \end{cases}$$

and $d_0(\sigma x) = *$ for x in K_0.

The following lemma is essential in showing that ev_0 is a weak equivalence.

Lemma 3.22. *Let $\mathcal{C}$ a path category and let $\varphi : Z \to \mathrm{ev}(X_\bullet)$ a morphism with $X_\bullet$ a frame. We can define a homotopical semisimplicial object $Y_\bullet$, a morphism $f_\bullet : Y_\bullet \to X_\bullet$ and a weak equivalence $w : Y_0 \to Z$ such that $\varphi \circ w = f_0$.*

Proof. By $P(n)$ we denote the semisimplicial set with precisely one i simplex for every $i \leq n$. Write $C(n)$ for the cone $CP(n)$. Note that we have the following chain of elementary expansions

$$\{*\} \to C(0) \to C(1) \to C(2) \to \cdots$$

Which means we the following chain of acyclic fibrations

$$\cdots \to C(2) \cap X_\bullet \to C(1) \cap X_\bullet \to C(0) \cap X_\bullet \to X_0.$$

Define Y_0 to be the following pullback

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] & C(0) \cap X_\bullet \arrow[d] \\
Z \arrow[r] & X_0,
\end{tikzcd}$$

and define Y_n to be this pullback

$$\begin{tikzcd}
Y_n \arrow[r] \arrow[d] & C(n) \cap X_\bullet \arrow[d] \\
Y_{n-1} \arrow[r] & C(n-1) \cap X_\bullet.
\end{tikzcd}$$

The map $Y_n \to Y_{n-1}$ is a weak equivalence for every n, hence we obtain a homotopical semisimplicial object $Y_\bullet$, with every face map in the same degree being the same. For every n there is a unique morphism $\Delta_+[n] \to P(n)$, identifying all i-simplices for $i \leq n$. We compose this map with the canonical inclusion $P(n) \to C(n)$ and obtain the following commutative diagram:

$$\begin{tikzcd}
\Delta_+[n-1] \arrow[r] \arrow[d] & P(n-1) \arrow[r] \arrow[d] & C(n-1) \arrow[d] \\
\Delta_+[n] \arrow[r] & P(n) \arrow[r] & C(n),
\end{tikzcd}$$

where we can range over all different inclusions $\Delta_+[n-1] \to \Delta[n]$. Applying $- \cap X_\bullet$ to the outer rectangle yields the commutative diagram

$$\begin{tikzcd}
C(n) \cap X_\bullet \arrow[r] \arrow[d] & X_n \arrow[d] \\
C(n-1) \cap X_\bullet \arrow[r] & X_{n-1},
\end{tikzcd}$$

where we again range over all different inclusions $\Delta_+[n-1] \to \Delta_+[n]$. We can conclude now by remarking that we obtain a morphism $f_\bullet : Y_\bullet \to X_\bullet$ by composing $Y_n \to C(n) \cap X_\bullet$ with $C(n) \cap X_\bullet \to X_n$. In degree zero this map is precisely given such that $\varphi \circ w = f_0$. $\square$

Theorem 3.23. *The functor* $\mathrm{ev}_0 : \mathrm{Fr}\,\mathcal{C} \to \mathcal{C}$ *is a weak equivalence.*

Proof. First remark that ev_0 reflects weak equivalences by repeatedly applying 2-out-of-3: suppose that $f_\bullet : Y_\bullet \to X_\bullet$ is a morphism of frames such that f_0 is a weak equivalence. Now apply the following inductive argument: if f_i is a weak equivalence, then f_{i+1} is one because $f_i d_k = d_k f_{i+1}$ and both face maps are weak equivalences.

We will now show the approximation property holds. Let $\varphi : Z \to \mathrm{ev}_0(X_\bullet)$ any morphism. Apply the previous lemma to obtain a triangle

$$\begin{tikzcd} \mathrm{ev}_0(Y_\bullet) \arrow[d, "w"'] \arrow[dr, "\mathrm{ev}_0(f_\bullet)"] & \\ Z \arrow[r, "\varphi"'] & \mathrm{ev}_0(X_\bullet). \end{tikzcd}$$

Note that $Y_\bullet$ is not necessarily fibrant. We will fix this by factorizing $f_\bullet$. First, factorize f_0 as follows

$$Y_0 \xrightarrow{v_0} Y_0' \xrightarrow{p_0} X_0,$$

with v_0 a weak equivalence and p_0 a fibration. Then define Y_n' as follows. First take the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X_n \arrow[d] \\ M_n Y_\bullet' \arrow[r] & M_n X_\bullet. \end{tikzcd}$$

By the universal property of the pullback we obtain a map $Y_n \to P$ which we then factorize as

$$Y_n \xrightarrow{v_n} Y_n' \longrightarrow P.$$

The map p_n now is the composition

$$Y_n' \to P \to X_n.$$

By induction it becomes clear that we have factorized $f_\bullet : Y_\bullet \to X_\bullet$ as

$$Y_\bullet \xrightarrow{v_\bullet} Y_\bullet' \xrightarrow{p} X_\bullet,$$

with $Y_\bullet'$ being fibrant, $v_\bullet$ a weak equivalence and $p_\bullet$ a fibration. By 2-out-of-3, and $Y_\bullet$ being homotopical, it follows that $Y_\bullet'$ is homotopical too. Thus we obtain the following commutative square in $\mathcal{C}$:

$$\begin{tikzcd} Y_0 \arrow[r, "v_0"] \arrow[d, "w"'] & \mathrm{ev}_0(Y_\bullet') \arrow[d, "\mathrm{ev}_0(p_\bullet)"] \\ Z \arrow[r, "\varphi"'] & \mathrm{ev}_0(X_\bullet), \end{tikzcd}$$

which shows that the approximation property holds, and hence ev_0 is a weak equivalence. $\square$

Theorem 3.24. *The functor* $\mathrm{ev}_0 : \mathrm{Fr}\,\mathcal{C} \to \mathcal{C}$ *is a fibration.*

Proof. We check the properties:

(i) ev_0 *is an isofibration:* let $i : X_0 \to Y$ an isomorphism. Suppose that we have extended i up until level $i_{n-1} : X_{n-1} \to Y_{n-1}$. By the theory on inverse diagrams, there is an induced isomorphism $M_n i_\bullet : M_n X_\bullet \to M_n Y_\bullet$, where $X_\bullet$ and $Y_\bullet$ are only defined up to level $n-1$. Now set $Y_n := X_n$ and define the map $Y_n \to M_n Y_\bullet$ to be $M_n i_\bullet \circ x_n$, where x_n denotes the map $X_n \to M_n X_\bullet$. We define i_n to be the identity. It is clear that this extends $i_\bullet$ another level.

(ii) ev_0 *has the lifting property for factorizations:* let $f_\bullet : X_\bullet \to Y_\bullet$ a morphism of frames and let $p_0 w_0$ a factorization of f_0. Suppose we have lifted the factorization up to degree $n-1$. Let $M_n Z_\bullet \times_{M_n Y_\bullet} Y_n$ the obvious pullback, and factorize $\langle \lim w_\bullet, f_n \rangle$ as a weak equivalence followed by a fibration. The object in the middle will be Z_n.

(iii) ev_0 *has the lifting property for sections of acyclic fibrations:* let $f_\bullet : X_\bullet \to Y_\bullet$ an acyclic fibration, and let s_0 a section of f_0. The argument is by induction, and in degree n obtained as follows. The map $\langle x_n, f_n \rangle : X_n \to M_n X_\bullet \times_{M_n Y_\bullet} Y_n$ is an acyclic fibration and hence has a section u. Now define $s_n := u \circ \langle \lim s_\bullet, id \rangle \circ y_n$. One can check that this constructs a section $s_\bullet$ of $f_\bullet$.

$\square$

3.5 The global sections functor

As we will see in Section 4.6 an important ingredient of the proof of canonicity in [11] is the *global sections functor*. For any category $\mathcal{C}$ with a terminal object this functor $\Gamma : \mathcal{C} \to \mathbf{Set}$ is defined for any object X as $\Gamma(X) = \mathrm{Hom}_{\mathcal{C}}(1, X)$. In this section we will be considering a version of the global sections functor which is valued in semisimplicial sets. Given a frame $X_\bullet$ in $\mathcal{C}$, we define $\Gamma(X_\bullet)$ to be the semisimplicial set for which the n-simplices are given by

$$\Gamma(X_\bullet)_n = \mathrm{Hom}_{\mathcal{C}}(1, X_n).$$

We will show that this functor is an exact functor of fibration categories. Let us first show that it is valued in fibrant semisimplicial sets:

Lemma 3.25. *Let* $\mathcal{C}$ *a path category and* $X_\bullet$ *a frame in* $\mathcal{C}$*. Then, the semisimplicial set* $\Gamma(X_\bullet)$ *is a semisimplicial Kan complex.*

Proof. Consider a diagram

$$\begin{array}{ccc} \Lambda^i_+[n] & \xrightarrow{\sigma} & \Gamma(X_\bullet) \\ \big\downarrow & & \\ \Delta_+[n] & & \end{array}$$

in **ssSet**. We want to find a lift $\widetilde{\sigma} : \Delta_+[n] \to \Gamma(X_\bullet)$ making the triangle commute. The horn σ in $\Gamma(X_\bullet)$ consists of a collection of $(n-1)$-simplices $x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ in $\Gamma(X_\bullet)$ such that they agree on the appropriate boundaries. These $(n-1)$-simplices are morphisms $x_k : 1 \to X_{n-1}$. We see that we get a cone on the diagram

$$\int \Lambda^i_+[n] \to \Delta_+ \to \mathcal{C}$$

and hence an induced map $\sigma' : 1 \to \Lambda^i_+[n] \cap X_\bullet$. Write π_k for the projection $\Lambda^i_+[n] \cap X_\bullet \to X_{n-1}$ corresponding to d_k. We have $\pi_k \sigma' = d_k$ for every $k \in \{0, \dots, x_{i-1}, x_{i+1}, \dots x_n\}$. Now we note that because $\Lambda^i_+[n] \to \Delta_+[n]$ is an elementary expansion, the map induced map $X_n \to \Lambda^i_+[n] \cap X_\bullet$ is an acyclic fibration. We now write m_x for this morphism. The morphism m_x moreover satisfies $\pi_k m_x = d_k$ for every relevant k. Let s a section of m_x. We claim that $s \circ \sigma' : 1 \to X_n$ is a horn filling n-simplex. To do this, it suffices to show that $d_k s\sigma' = x_k$ for relevant k. We get

$$\begin{aligned} d_k s\sigma' &= \pi_k m_x s\sigma' \\ &= \pi_k \sigma' \\ &= x_k, \end{aligned}$$

and conclude that $\Gamma(X_\bullet)$ is fibrant. □

Lemma 3.26. *Let $\mathcal{C}$ a path category and $f_\bullet : X_\bullet \to Y_\bullet$ a fibration in $\mathrm{Fr}\,\mathcal{C}$. Then, the morphism $\Gamma(f_\bullet) : \Gamma(X_\bullet) \to \Gamma(Y_\bullet)$ is a semisimplicial Kan fibration.*

Proof. Consider a diagram

$$\begin{tikzcd} \Lambda^k_+[n] \arrow[r, "\alpha"] \arrow[d] & \Gamma(X_\bullet) \arrow[d, "\Gamma(f_\bullet)"] \\ \Delta_+[n] \arrow[r, "\beta"'] & \Gamma(Y_\bullet) \end{tikzcd}$$

in **ssSet**. This corresponds to a diagram

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Lambda^k_+[n] \cap X_\bullet \arrow[d, "\Lambda^k_+[n] \cap f_\bullet"] \\ Y_n \arrow[r] & \Lambda^k_+[n] \cap Y_\bullet \end{tikzcd}$$

in $\mathcal{C}$. Write m_x for the map $X_n \to \Lambda^k_+[n] \cap X_\bullet$, similar for m_y. We are looking for a map $\gamma : 1 \to X_n$ such that both

$$\begin{tikzcd} & X_n \arrow[d, "m_x"] \\ 1 \arrow[ur, "\gamma"] \arrow[r, "\alpha"'] & \Lambda^k_+[n] \cap X_\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X_n \arrow[d, "f_n"] \\ 1 \arrow[ur, "\gamma"] \arrow[r, "\beta"'] & Y_n \end{tikzcd}$$

commute. Consider the pullback

$$\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y_n \arrow[d] \\
\Lambda^k_+[n] \cap X_\bullet \arrow[r] & \Lambda^k_+[n] \cap Y_\bullet.
\end{tikzcd}$$

We have an induced map $\gamma' := \langle \alpha, \beta \rangle : 1 \to P$. We also have an induced map $u : X_n \to P$. Because the maps $X_n \to \Lambda^k_+[n] \cap X_\bullet$ and $P \to \Lambda^k_+[n] \cap X_\bullet$ both are acyclic fibrations, the latter being a pullback of one, the map $X_n \to P$ is a weak equivalence. We will now show that the map is also a fibration. Since we have a factorization

$$Y_n \to M_n Y_\bullet \to \Lambda^k_+[n] \cap Y_\bullet$$

we can write the pullback as

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & Y_n \arrow[d] \\
P' \arrow[r] \arrow[d] & M_n Y_\bullet \arrow[d] \\
\Lambda^k_+[n] \cap X_\bullet \arrow[r] & \Lambda^k_+[n] \cap Y_\bullet,
\end{tikzcd}$$

by which we can see that the map $X_n \to P$ factorizes in the following way:

$$\begin{tikzcd}
X_n \arrow[dr] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& P'' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Y_n \arrow[d] \\
& M_n X_\bullet \arrow[r] & P' \arrow[r] \arrow[d] & M_n Y_\bullet \arrow[d] \\
& & \Lambda^k_+[n] \cap X_\bullet \arrow[r] & \Lambda^k_+[n] \cap Y_\bullet,
\end{tikzcd}$$

where the map $X_n \to P''$ is a fibration because $f_\bullet$ is a fibration. It remains to show that $P'' \to P$ is a fibration, which we will show by showing that $M_n X_\bullet \to P'$ is a fibration. Since $M_n Y_\bullet \to \Lambda^k_+[n] \cap Y_\bullet$ is the pullback of $Y_{n-1} \to M_{n-1} Y_\bullet$ we get the following pullback diagram

$$\begin{tikzcd}
P' \arrow[r] \arrow[d] & M_n Y_\bullet \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
\Lambda^k_+[n] \cap X_\bullet \arrow[r] & \Lambda^k_+[n] \cap Y_\bullet \arrow[r] & M_{n-1} Y_\bullet.
\end{tikzcd}$$

Note that the following diagram commutes

$$\begin{tikzcd}
\Lambda^k_+[n] \cap X_\bullet \arrow[r] \arrow[d] & \Lambda^k_+[n] \cap Y_\bullet \arrow[d] \\
M_{n-1}X_\bullet \arrow[r] & M_{n-1}Y_\bullet,
\end{tikzcd}$$

so we can also obtain P' as the following pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & P''' \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
\Lambda^k_+[n] \cap X_\bullet \arrow[r] & M_{n-1}X_\bullet \arrow[r] & M_{n-1}Y_\bullet.
\end{tikzcd}$$

Since $M_nX_\bullet \to \Lambda^k_+[n] \cap X_\bullet$ is a pullback of $X_{n-1} \to M_{n-1}X_\bullet$ we can see the map $M_nX \to P'$ arising as a pullback as in the following diagram

$$\begin{tikzcd}
M_nX_\bullet \arrow[r] \arrow[d] & X_{n-1} \arrow[d] & \\
P' \arrow[r] \arrow[d] & P''' \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
\Lambda^k_+[n] \cap X_\bullet \arrow[r] & M_{n-1}X_\bullet \arrow[r] & M_{n-1}Y_\bullet,
\end{tikzcd}$$

where the map $X_{n-1} \to P'''$ is a pullback by the fact that f is a pullback. This shows that $M_nX_\bullet \to P'$ is a pullback and hence u is an acyclic fibration with some section s. We now claim that $\gamma := s \circ \gamma'$ makes the things we want commute.

$$\begin{aligned}
m_x \circ \gamma &= m_x \circ s \circ \gamma' \\
&= \pi_1 \circ u \circ s \circ \gamma' \\
&= \pi_1 \circ \gamma' \\
&= \alpha
\end{aligned}$$

and

$$\begin{aligned}
f_n \circ \gamma &= f_n \circ s \circ \gamma' \\
&= \pi_2 \circ u \circ s \circ \gamma' \\
&= \pi_2 \circ \gamma' \\
&= \beta
\end{aligned}$$

show that this is indeed the case. This finishes the proof. $\square$

Lemma 3.27. *Let $\mathcal{C}$ a path category and $f_\bullet : X_\bullet \to Y_\bullet$ an acyclic fibration in* $\mathrm{Fr}\,\mathcal{C}$. *Then, the morphism $\Gamma(f_\bullet)$ is a trivial fibration.*

Proof. Consider a commutative square

$$\begin{tikzcd}
\partial\Delta^n_+ \arrow[r] \arrow[d] & \Gamma(X_\bullet) \arrow[d] \\
\Delta^n_+ \arrow[r] & \Gamma(Y_\bullet)
\end{tikzcd}$$

in **ssSet**. This corresponds to a commutative square

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & M_n X_\bullet \arrow[d] \\
Y_n \arrow[r] & M_n Y_\bullet
\end{tikzcd}$$

in $\mathcal{C}$. Let

$$\begin{tikzcd}
M_n X_\bullet \times_{M_n Y_\bullet} Y_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\
M_n X_\bullet \arrow[r] & M_n Y_\bullet
\end{tikzcd}$$

a pullback diagram. We have an induced map $\gamma' := \langle \alpha, \beta \rangle : 1 \to M_n X_\bullet \times_{M_n Y_\bullet} Y_n$. We also have an induced map $X_n \to M_n X_\bullet \times_{M_n Y_\bullet} Y_n$, which is a fibration because $f_\bullet$ is a fibration. This map is moreover a weak equivalence by 2-out-of-3; because $M_n X_\bullet \times_{M_n Y_\bullet} Y_n \to Y_n$ and $X_n \to Y_n$ are weak equivalences. This means we can get a section $s : M_n X_\bullet \times_{M_n Y_\bullet} Y_n \to X_n$, and obtain a filler $\gamma := s \circ \gamma'$. □

Corollary 3.28. *Let $\mathcal{C}$ a path category and $f_\bullet : X_\bullet \to Y_\bullet$ a weak equivalence in* $\mathrm{Fr}\,\mathcal{C}$*. Then $\Gamma(f_\bullet)$ is a weak equivalence.*

Proof. We factorize $f_\bullet$ as a section $w_{f_\bullet} : X_\bullet \to Z_\bullet$ of an acyclic fibration $g_\bullet : Z_\bullet \to X_\bullet$ followed by an acyclic fibration $p_{f_\bullet} : Z_\bullet \to Y_\bullet$. The maps $\Gamma(g_\bullet)$ and $\Gamma(p_{f_\bullet})$ are acyclic fibrations by Lemma 3.27. By 2-out-of-3 the map $\Gamma(w_{f_\bullet})$ is one, and hence $\Gamma(f_\bullet)$ is. □

We can now prove the main theorem of this section.

Theorem 3.29. *Let $\mathcal{C}$ a path category. The functor $\Gamma : \mathrm{Fr}\,\mathcal{C} \to$* **ssSet** *yields an exact functor $\mathrm{Fr}\,\mathcal{C} \to$* $\mathbf{ssSet}_\mathrm{f}$ *of path categories.*

Proof. We have shown that Γ preserves fibrations. It is moreover clear that the terminal object is preserved. Since pullbacks in both $\mathrm{Fr}\,\mathcal{C}$ and **ssSet** are pointwise, and the hom functor preserves limits we have that a pullback $X_\bullet \times_{Y_\bullet} \times Z_\bullet$ in $\mathrm{Fr}\,\mathcal{C}$ with components $X_n \times_{Y_n} Z_n$ is mapped to the pullback $\Gamma(X_\bullet) \times_{\Gamma(Y_\bullet)} \Gamma(Z_\bullet)$ with components $\mathrm{Hom}(1, X_n) \times_{\mathrm{Hom}(1,Y_n)} \times \mathrm{Hom}(1, Z_n)$. We conclude that the functor is exact. □

3.6 Enrichment over semisimplicial sets

In [15] and [14], it is shown that the fibration category of frames in a fibration category always can be canonically enriched over semisimplicial sets. We will briefly summarize the construction, and state without proof that this also holds true for path categories. We moreover show that in the case of path categories all hom-semisimplicial sets are fibrant.

Recall the pairing $K \pitchfork X_\bullet$ of a finite semisimplicial set and a frame in $\mathcal{C}$ as defined in Section 3.3. We will now extend this pairing in such a way that given finite K and a frame $X_\bullet$ the result $X_\bullet^K$ is a frame. In degree n the frame is defined as

$$(X_\bullet^K)_n := (\Delta_+[n] \otimes K) \pitchfork X_\bullet.$$

This operation will be the *cotensor* in the enrichment, and is functorial in both arguments. By fixing $X_\bullet$, we obtain a functor $X_\bullet^{(-)}$ which maps colimits to limits because limits in $\operatorname{Fr}\mathcal{C}$ are pointwise and the functor $- \otimes \Delta_+[n]$ preserves colimits because it is a left adjoint.

Let us now define the semisimplicial set $\operatorname{Fr}\mathcal{C}(X_\bullet, Y_\bullet)$ corresponding to the morphisms between two frames $X_\bullet$ and $Y_\bullet$. In degree n this is defined as

$$\operatorname{Fr}\mathcal{C}(X_\bullet, Y_\bullet)_n := \operatorname{Hom}_{\operatorname{Fr}\mathcal{C}}(X_\bullet, Y_\bullet^{\Delta_+[n]}).$$

That the enrichment is cotensored means that there is a natural isomorphism

$$\operatorname{Fr}\mathcal{C}(X_\bullet, Y_\bullet^K) \cong [K, \operatorname{Fr}\mathcal{C}(X_\bullet, Y_\bullet)]$$

where the right hand side denotes the internal hom of the symmetric monoidal structure on **ssSet**.

This enrichment satisfies the so-called *pullback-cotensor property.* This states that for a monomorphism $i : K \to L$ and a fibration $f_\bullet : X_\bullet \to Y_\bullet$ the induced map

$$X_\bullet^L \to X_\bullet^K \times_{Y_\bullet^K} Y_\bullet^L$$

is a fibration, which is an acyclic fibration is either $f_\bullet$ or i is a weak equivalence. In particular this means that for a monomorphism $i : K \to L$ and a frame $X_\bullet$ the induced map $X_\bullet^i : X_\bullet^L \to X_\bullet^K$ is a fibration by applying the pullback-cotensor property to the monomorphism i and the fibration $X_\bullet \to 1$. We summarize this in the following definition of a semisimplicial path category:

Definition 3.30. Let $\mathcal{C}$ a path category. We call $\mathcal{C}$ a semisimplicial path category if it carries a semisimplicial enrichment giving the ordinary homsets in degree 0, is cotensored over finite semisimplicial sets, and satisfies the pullback-cotensor property.

Theorem 3.31. *Let $\mathcal{C}$ a path category. The category $\operatorname{Fr}\mathcal{C}$ of frames in $\mathcal{C}$ is a semisimplicial path category.*

Proof. Theorem 3.10 and 3.17 in [14]. $\square$

The category of semisimplicial path categories and exact functors preserving the enriched structure form a fibration category:

Theorem 3.32. *The category of semisimplicial path categories and semisimplicial exact functors preserving finite cotensors form a fibration category, where the weak equivalences are given by weak equivalences on underlying path categories, and fibrations are given by fibrations on underlying path categories.*

Proof. Theorem 4.9 in [15]. □

Remark 3.33. We mentioned in the beginning of Section 2.4 that Kapulkin and Szumilo in their paper [15] compare the homotopy theory of fibration categories and tribes. It turns out that the category of tribes do not form a fibration category, but that the category of semisimplicial tribes **ssTrb** do form a fibration category, which is weakly equivalent to the fibration category of semisimplicial fibration categories **ssTrb**. However, the category of tribes **Trb** can be endowed with the structure of what is called a *homotopical category* in which only a class of weak equivalences exists. As homotopical categories, the categories **Trb** and **ssTrb** are weakly equivalent. Similarly for **ssFib** and **Fib**. Since weak equivalences between homotopical categories satisfy 2-out-of-3 and by considering the following diagram:

$$\begin{array}{ccc} \mathbf{ssTrb} & \longrightarrow & \mathbf{ssFib} \\ \downarrow & & \downarrow \\ \mathbf{Trb} & \longrightarrow & \mathbf{Fib} \end{array}$$

it follows that **Trb** and **Fib** are weakly equivalent

Theorem 3.34. *For any two frames $X_\bullet$ and $Y_\bullet$ in some path category $\mathcal{C}$, the semisimplicial set $\mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)$ is a semisimplicial Kan complex.*

Proof. Let $i : \Lambda^k_+[n] \to \Delta_+[n]$ a horn inclusion and let $X_\bullet$ and $Y_\bullet$ frames in a path category $\mathcal{C}$. We want to show that the induced map

$$\mathrm{Hom}_{\mathbf{ssSet}}(\Delta_+[n], \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)) \to \mathrm{Hom}_{\mathbf{ssSet}}(\Lambda^k_+[n], \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet))$$

is a surjection. By the Yoneda lemma the domain of the map above corresponds to

$$\mathrm{Hom}_{\mathrm{Fr}\,\mathcal{C}}(X_\bullet, Y_\bullet^{\Delta_+[n]}).$$

The codomain corresponds to

$$\mathrm{Hom}_{\mathrm{Fr}\,\mathcal{C}}(X_\bullet, Y_\bullet^{\Lambda^k_+[n]})$$

by the following chain of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{ssSet}}(\Lambda^k_+[n], \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)) &\cong \mathrm{Hom}_{\mathbf{ssSet}}(\Lambda^k_+[0], [\Delta[n], \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)]) \\ &\cong \mathrm{Hom}_{\mathbf{ssSet}}(\Lambda^k_+[0], \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet^{\Lambda^k_+[n]})) \\ &\cong \mathrm{Hom}_{\mathrm{Fr}\,\mathcal{C}}(X_\bullet, Y_\bullet^{\Lambda^k_+[n]}). \end{aligned}$$

Hence the map induced by the horn inclusion i corresponds to the map

$$\mathrm{Hom}_{\mathrm{Fr}\,\mathcal{C}}(X_\bullet, Y_\bullet^{\Delta_+[n]}) \to \mathrm{Hom}_{\mathrm{Fr}\,\mathcal{C}}(X_\bullet, Y_\bullet^{\Lambda_+^k[n]}),$$

mapping a morphism $f_\bullet : X_\bullet \to Y_\bullet^{\Delta_+[n]}$ to $Y_\bullet^i f_\bullet$. Because i is a monomorphism the map $Y_\bullet^i$ is a fibration by the pullback-cotensor property. It moreover is a weak equivalence since $(Y_\bullet^i)_0$ is a weak equivalence because i is an elementary expansion, and ev_0 reflects weak equivalences. We conclude it is an acyclic fibration and hence has a section $s_\bullet : Y_\bullet^{\Lambda_+^k[n]} \to Y_\bullet^{\Delta_+[n]}$. We can now conclude that we have surjectivity, because for any $g_\bullet : X_\bullet \to Y_\bullet^{\Lambda_+^k[n]}$ the map $s_\bullet g_\bullet$ is mapped to $g_\bullet$. $\square$

Remark 3.35. One might be tempted to describe conclude from Theorem 3.34 that "path categories are enriched over fibrant semisimplicial sets". However, as $\mathbf{ssSet}_\mathrm{f}$ is not closed under the geometric product, as discussed in [35], this statement does not make sense.

3.7 Enrichment over groupoids

Any path category has a natural enrichment over groupoids, as introduced by Den Besten in [16]. In this section we will review this enrichment. Moreover, we will describe the fibration category structure of the category **GpdCat** obtained by Lack's model structure on **GpdCat** as described in [36] and [37]. We will show that the canonical functor $M : \mathbf{Pth} \to \mathbf{GpdCat}$ which maps a path category to an object in **GpdCat** preserves fibrations, weak equivalences and the terminal object.

Let X, Y two objects in a path category $\mathcal{C}$. We define the groupoid $\mathcal{C}(X, Y)$ as the category which has as objects the morphisms $f : X \to Y$, and as morphisms equivalence classes of homotopies $H : X \to PY$ identified up to fibrewise homotopy over $Y \times Y$. In [16] it is shown that this enrichment is independent of the choice of path object.

Theorem 3.36. *The enrichment as described above defines a 2-functor $M : \mathbf{Pth} \to \mathbf{GpdCat}$.*

Proof. Section 3 in [16]. $\square$

We will not use the higher categorical structure and only focus on the ordinary functor $M : \mathbf{Pth} \to \mathbf{GpdCat}$.

Let us now turn to the model structure on **GpdCat**. Given a 2-category $\mathcal{C}$, a 1-cell $f : X \to Y$ is an *equivalence* if there is a 1-cell $g : Y \to X$ such that fg and gf are isomorphic to the identities. One should note that for the enrichment of path categories, these equivalences are precisely the weak equivalences.

> **Weak equivalences** A strict functor $F : \mathcal{C} \to \mathcal{D}$ between two 2-categories is a weak equivalence if it is *biessentially surjective*: for every 0-cell X in $\mathcal{D}$ there is some Y in $\mathcal{C}$ and an equivalence $f : FY \to X$. It should moreover be *locally an equivalence*. That is, for every two 0-cells A and B in $\mathcal{C}$ the functor $\mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ should be an equivalence of categories.

Fibrations A strict functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories is a fibration if every equivalence $Y \to FX$ in $\mathcal{D}$ can be lifted to an equivalence $Y' \to X$ in $\mathcal{C}$. Moreover, we need that every invertible 2-cell $H : f \to Fg$ in $\mathcal{D}$ should be lifted to an invertible 2-cell $H' : f' \to g$ in $\mathcal{C}$.

Cofibrations A strict functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories is a cofibration if it has the left lifting property w.r.t acyclic fibrations.

The acyclic fibrations in this model structure are the functors which are surjective on objects and which are locally surjective on objects and an equivalence.

We see that our definition of a 2-fibration (Definition 2.50) precisely coincides with the fibrations of path categories which are mapped to a fibration of 2-categories by the functor M, and hence we obtain the following lemma.

Lemma 3.37. *The functor M : **Pth** $\to$ **GpdCat** preserves fibrations.*

Lemma 3.38. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration of path categories. Then $F_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ is an equivalence of categories.*

Proof. We show that $F_{X,Y}$ is essentially surjective, full and faithful. Since F is full, the functor $F_{X,Y}$ is surjective and in particular essentially surjective. For fullness of $F_{X,Y}$ suppose that $H : FX \to F(PY)$ is a homotopy between maps $F(f)$ and $F(g)$. We factorize (f,g) as a weak equivalence $w : X \to T$ followed by a fibration $(p_f,p_g) : T \to Y \times Y$. We obtain the following diagram in $\mathcal{D}$:

$$\begin{tikzcd} FX \arrow[r, "H"] \arrow[d, "Fw"'] & F(PY) \arrow[d, "F(s,t)"] \\ FT \arrow[r, "F(p_f,p_g)"'] & F(Y \times Y), \end{tikzcd}$$

for which there is a diagonal filler $H' : FT \to F(PY)$. Because the functor $F(Y \times Y)$ is an acyclic fibration and hence full, we can obtain a morphism $\widetilde{H}' : T \to PY$ which is mapped to H' by F. The morphism $\widetilde{H}'$ is a homotopy between p_f and p_g in $\mathcal{C}$. Let us now consider the map $\widetilde{H} := \widetilde{H}'w : X \to PY$. This map is a homotopy between f and g, and is mapped to $H'F(w)$ by F. The latter is fibrewise homotopic to H, which concludes that $F_{X,Y}$ is full. Let us now show that $F_{X,Y}$ is faithful. Suppose we have two homotopies H, H' between maps f and g such that $F(H) \simeq_{F(Y \times Y)} F(H')$. It follows by Lemma 2.49 that H and H' and H are fibrewise homotopic over $Y \times Y$ and hence represent the same equivalence class of homotopies between f and g. ☐

Lemma 3.39. *The functor M : **Pth** $\to$ **GpdCat** preserves acyclic fibrations*

Proof. This follows from Lemma 3.38 and the fact that acyclic fibrations are full and surjective on objects. ☐

Corollary 3.40. *The functor M : **Pth** $\to$ **GpdCat** preserves weak equivalences.*

Remark 3.41. It is an open question whether the functor $M : \mathbf{Pth} \to \mathbf{GpdCat}$ preserves pullbacks along fibrations. Given an exact functor $F : \mathcal{C} \to \mathcal{D}$ and a fibration $G : \mathcal{E} \to \mathcal{D}$ in $\mathbf{Pth}$, one has to compare the groupoid enriched categories $M(\mathcal{C}) \times_{\mathcal{M}(\mathcal{D})} M(\mathcal{E})$ and $M(\mathcal{C} \times_{\mathcal{D}} \mathcal{E})$. By the universal property of the pullback there is a unique map $\Psi : M(\mathcal{C} \times_{\mathcal{D}} \mathcal{E}) \to M(\mathcal{C}) \times_{M(\mathcal{D})} M(\mathcal{E})$ in $\mathbf{GpdCat}$. As pullbacks in $\mathbf{GpdCat}$ are computed pointwise, it follows immediately that on 0- and 1-cells the map Ψ is an isomorphism. The uncertainty of a hypothetical proof lies in showing that the component Ψ_2 is injective; showing that Ψ_2 is well-defined and surjective can be done by the homotopy lifting properties of the functor $\pi_0 : \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to \mathcal{C}$ and $\pi_{0/(Y,Y') \times (Y,Y')} : (\mathcal{C} \times_{\mathcal{D}} \mathcal{E}_{/(Y,Y') \times (Y,Y')})_{\mathrm{f}} \to (\mathcal{C}_{/Y})_{\mathrm{f}}$. Showing that Ψ_2 is injective amounts to showing that two pairs of homotopies (H, H') and (K, K') in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ between two pairs of maps (f, f') and (g, g') are identified if and only if they are pointwise identified. As

$$(\mathcal{C} \times_{\mathcal{D}} \mathcal{E}_{/(Y,Y') \times (Y,Y')})_{\mathrm{f}}$$

is isomorphic to

$$(\mathcal{C}_{/Y})_{\mathrm{f}} \times_{(\mathcal{D}_{/F(Y)})_{\mathrm{f}}} (\mathcal{E}_{/Y'})_{\mathrm{f}}$$

this will follow immediately if one can show that the homotopy relation in a pullback in $\mathbf{Pth}$ is determined pointwise.

3.8 Fundamental groupoid

Given a path category $\mathcal{C}$, we have seen that there are two canonical enrichments of the category $\mathrm{Fr}\,\mathcal{C}$: one over groupoids and one over semisimplicial sets. In this section we will compare both enrichments by means of the *fundamental groupoid*. The fundamental groupoid was originally defined for a topological space X to be the groupoid consisting of the points of the spaces and equivalence classes of paths identified up to endpoint preserving homotopy.

One can make an educated guess and try to define a semisimplicial version of the fundamental groupoid by taking as objects the vertices and as morphisms the 1-cells identified up to some notion of endpoint preserving semisimplicial homotopy. It turns out, as is shown in Section 3.3 of [33], that this construction is well defined if the semisimplicial set in consideration is fibrant.

In Theorem 3.34 we have shown that for two frames $X_\bullet$ and $Y_\bullet$ in some category $\mathcal{C}$, the semisimplicial set $\mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)$ is a semisimplicial Kan complex. In Theorem 3.45 we will show that applying the fundamental groupoid to $\mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)$, we obtain the same groupoid as the one you get by considering the canonical enrichment of $\mathrm{Fr}\,\mathcal{C}$ over groupoids.

Before we can define the semisimplicial fundamental groupoid we observe that we can obtain a semisimplicial version of the sphere S^1 as the following pushout:

$$\begin{tikzcd}
\Delta_+[0] \coprod \Delta_+[0] \arrow[r] \arrow[d] & \Delta_+[1] \arrow[d] \\
\Delta_+[1] \arrow[r] & S^1.
\end{tikzcd}$$

Definition 3.42. Let X a semisimplicial Kan complex. The *fundamental groupoid* $\Pi(X)$ has objects the vertices $\Delta_+[0] \to X$ and as morphisms between two such vertices $x, y : \Delta_+[0] \to X$, we have 1-cells $\omega : \Delta_+[1] \to X$ between x and y identified up to *relatedness.* Two 1-cells are related if the induced map $\langle \omega, \gamma \rangle : S^1 \to X$ factorises through some homotopy $H : CS^1 \to X$, as in the following diagram:

$$\begin{tikzcd} & CS^1 \arrow[d, "H"] \\ S^1 \arrow[ur] \arrow[r, "\langle \omega,\gamma \rangle"'] & X, \end{tikzcd}$$

where $S^1 \to CS^1$ denotes the canonical inclusion.

Lemma 3.43. *Let $Y_\bullet$ a frame in $\mathcal{C}$. The cotensor $Y_\bullet^{\Delta_+[1]}$ is a path object on $Y_\bullet$.*

Proof. We note that because the map $\partial\Delta_+[1] \to \Delta_+[1]$ is a inclusion, the induced map $X_\bullet^{\Delta_+[1]} \to X_\bullet^{\partial\Delta_+[1]}$ is a fibration. Also note that $X_\bullet^{\partial\Delta_+[1]} = X_\bullet \times X_\bullet$ because the pairing $- \pitchfork X_\bullet$ maps colimits to limits. The components of the fibration $X_\bullet^{\Delta_+[1]} \to X_\bullet^{\partial\Delta_+[1]}$ are acyclic fibrations because they are induced by inclusions $\Delta_+[0] \to \Delta_+[1]$ which are weak equivalences of semisimplicial sets. In degree 0, the morphism $X_\bullet^{\Delta_+[1]} \to X_\bullet^{\partial\Delta_+[1]}$ coincides with the map $X_1 \to X_0 \times X_0$. In this map, both components are homotopic by Lemma 3.3. Because ev_0 is homotopy faithful, it follows that both components of the map $X_\bullet^{\Delta_+[1]} \to X_\bullet \times X_\bullet$ are homotopic. It follows by Lemma 2.24 that the components have a common section and hence $X_\bullet^{\Delta_+[1]}$ is a path object. $\square$

Lemma 3.44. *Let $Y_\bullet$ a frame in $\mathcal{C}$. Let CS^1 the cone on S^1. Then the cotensor $Y_\bullet^{CS^1}$ is the fibred path object of the fibration $Y_\bullet^{\Delta_+[1]} \to Y_\bullet \times Y_\bullet$.*

Proof. Recall that $Y_\bullet^{S^1}$ is isomorphic to the pullback $Y_\bullet^{\Delta_+[1]} \times_{Y_\bullet \times Y_\bullet} Y_\bullet^{\Delta_+[1]}$. The inclusion $S^1 \to CS^1$ induces a fibration $Y_\bullet^{CS^1} \to Y_\bullet^{S^1}$. We want to show that this fibration fits into a factorization
$$Y_\bullet^{\Delta_+[1]} \to Y_\bullet^{CS^1} \to Y_\bullet^{S^1}$$
of the diagonal on $Y_\bullet^{\Delta_+[1]}$ in $(\mathrm{Fr}\,\mathcal{C}_{/Y_\bullet \times Y_\bullet})_{\mathrm{f}}$. The map $Y_\bullet^{CS^1} \to Y_\bullet^{S^1}$ has two components $Y_\bullet^{CS^1} \to Y_\bullet^{\Delta_+[1]}$ which are induced by the two inclusions $\Delta_+[1] \to CS^1$ which factor through S^1. One can write these inclusions as compositions of elementary expansions, and hence they induce acyclic fibrations $Y_\bullet^{CS^1} \to Y_\bullet^{\Delta_+[1]}$. By Lemma 2.24 it is sufficient to show that the components $Y_\bullet^{CS^1} \to Y_\bullet^{\Delta_+[1]}$ are fibrewise homotopic over $Y_\bullet^{\Delta_+[1]} \to Y_\bullet \times Y_\bullet$ to make them fit into a fibrewise path object. Because ev_0 is a weak equivalence it is sufficient to show that the maps are fibrewise homotopic when evaluated at 0, by Lemma 2.49. Observe that the cone CS^1 is isomorphic to the following pushout:

$$\begin{tikzcd} \Lambda^0_+[2] \arrow[r] \arrow[d] & \Delta_+[2] \arrow[d] \\ \Delta_+[2] \arrow[r] & CS^1. \end{tikzcd}$$

By applying $- \cap Y_\bullet$ we obtain the pullback

$$\begin{tikzcd} CS^1 \cap Y_\bullet \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & Y_2 \arrow[d] \\ Y_2 \arrow[r] & \Lambda^0_+[2] \cap Y_\bullet, \end{tikzcd}$$

in which every map is an acyclic fibration because $\Lambda^0_+[2] \to \Delta_+[2]$ is an elementary expansion. The inclusions $\Delta_+[1] \to CS^1$ we are interested in factor through the inclusion $\Delta_+[2] \to CS^1$. In particular this means that we obtain the induced components $CS^1 \cap Y_\bullet \to Y_1$ as the maps $d_2\pi_0$ and $d_2\pi_1$. The maps coincide on the map $Y_2 \to \Lambda^0_+[2] \cap Y_\bullet$ and hence are fibrewise homotopic by Lemma 2.15. Observe that the following diagram commutes:

$$\begin{tikzcd} Y_2 \arrow[r, "d_2"] \arrow[d] & Y_1 \arrow[d] \\ \Lambda^0_+[2] \cap Y_\bullet \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

and hence we can apply Lemma 2.17 to conclude that the two inclusions are fibrewise homotopic and hence that $Y_\bullet^{CS^1}$ is a fibrewise path object on the fibration $Y_\bullet^{\Delta_+[1]} \to Y_\bullet^{\partial\Delta_+[1]}$. □

Theorem 3.45. *Let $\mathcal{C}$ a path category and $X_\bullet$ and $Y_\bullet$ two frames in $\mathcal{C}$. The fundamental groupoid $\Pi(\mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet))$ is naturally isomorphic to the groupoid $M(\mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet))$.*

Proof. The objects of the fundamental groupoid $\Pi(\mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet))$ are the vertices $\Delta_+[0] \to \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)$, i.e. just the ordinary morphisms $X_\bullet \to Y_\bullet$. Morphisms between two vertices $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ are 1-cells $\omega : \Delta_+[1] \to \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)$ between $f_\bullet$ and $g_\bullet$, identified up to relatedness. A 1-cell $\Delta_+[1] \to \mathrm{Fr}\,\mathcal{C}(X_\bullet, Y_\bullet)$ corresponds to a morphism $X_\bullet \to Y_\bullet^{\Delta_+[1]}$, whereas the codomain of this map is a path object on $Y_\bullet$ by Lemma 3.43. It remains to show that relatedness coincides with fibrewise homotopy over $Y_\bullet \times Y_\bullet$.

Suppose two 1-cells $\omega_\bullet, \gamma_\bullet : X_\bullet \to Y_\bullet^{\Delta_+[1]}$ between vertices $f_\bullet$ and $g_\bullet$ are related. Then the map $\langle \omega, \gamma \rangle : X_\bullet \to Y_\bullet^{\Delta_+[1]} \times_{Y_\bullet \times Y_\bullet} Y_\bullet^{\Delta_+[1]}$ factor through $Y_\bullet^{CS^1}$, which is precisely the fibrewise path object by Lemma 3.44. Now suppose two 1-cells $\omega_\bullet, \gamma_\bullet$ are fibrewise homotopic over $Y_\bullet \times Y_\bullet$. Because fibrewise homotopy is independent of the choice of path object there is a homotopy $X_\bullet \to Y_\bullet^{CS^1}$ such that the composition

$$X_\bullet \to Y_\bullet^{CS^1} \to Y_\bullet^{\Delta_+[1]} \times_{Y_\bullet \times Y_\bullet} Y_\bullet^{\Delta_+[1]}$$

precisely is $\langle \omega_\bullet, \gamma_\bullet \rangle$ and hence they are related. This finishes the proof. □

4 Natural numbers, function spaces and universes

In this chapter we will discuss the homotopy universal constructions which are relevant for the version of objective type theory we are considering. The first five sections will all have the same structure: we introduce an object with some homotopy universal property in a path category, and then show that this property is invariant under weak equivalences of path categories. In these proofs we will heavily rely on the properties of acyclic fibrations we have proven in Chapter 2.

In Section 4.6 we will informally discuss the proof for "ordinary" canonicity for intuitionistic type theory, and the hypothetical proof of homotopy canonicity for objective type theory.

4.1 Homotopy natural numbers objects

The first objects we will be considering are the so-called homotopy natural numbers objects. These are the categorical interpretation of the natural numbers type in objective type theory.

Definition 4.1. A *homotopy natural numbers object (hnno)* in a path category $\mathcal{C}$ consists of an object $\mathbb{N}$ and maps $0 : 1 \to \mathbb{N}$ and $S : \mathbb{N} \to \mathbb{N}$ such that for every commuting diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x_0"] \arrow[r, "0"'] & \mathbb{N} \arrow[r, "S"'] & \mathbb{N} \end{tikzcd}$$

with p a fibration, there exists a section $a : \mathbb{N} \to P$ of p, such that $a0 \simeq_{\mathbb{N}} x_0$ and $aS \simeq_{\mathbb{N}} fa$.

Lemma 4.2. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. If $\mathcal{C}$ has a hnno, then F preserves it.*

Proof. Suppose $(\mathbb{N}, 0, S)$ is a hnno in $\mathcal{C}$. Suppose we have a commutative diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x_0"] \arrow[r, "F(0)"'] & F(\mathbb{N}) \arrow[r, "F(S)"'] & F(\mathbb{N}) \end{tikzcd}$$

in $\mathcal{D}$. Use the characterization of acylic fibrations to lift the fibration $p : X \to F(\mathbb{N})$ to a fibration $p' : X' \to \mathbb{N}$. We use Lemma 2.56 to lift $x_0 : 1 \to X$ to a map x_0', similarly we lift $f : X \to X$ to a map $f' : X' \to X'$. Because $(\mathbb{N}, 0, S)$ is a hnno we obtain a section $a : \mathbb{N} \to X'$ of p'. Now the map $F(a)$ is a section of p, and because F preserves fibrewise homotopies we have that $F(a)F(0) \simeq_{F(\mathbb{N})} F(x'_0)$ and $F(a)F(0) \simeq_{F(\mathbb{N})} F(f')F(a)$ because $F(x'_0) = x_0$ and $F(f') = f$. □

Lemma 4.3. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let $(\mathbb{N}, 0, S)$ a hnno in $\mathcal{D}$. If $(\mathbb{N}', 0', S')$ is mapped to $(\mathbb{N}, 0, S)$ by F then $(\mathbb{N}', 0', S')$ is a hnno.*

Proof. Suppose we have a commutative diagram

$$\begin{tikzcd} & X' \arrow[r, "f'"] \arrow[d, "p'"] & X' \arrow[d, "p'"] \\ 1 \arrow[ur, "x'_0"] \arrow[r, "0'"'] & \mathbb{N}' \arrow[r, "S'"'] & \mathbb{N}' \end{tikzcd}$$

in $\mathcal{C}$. It is mapped to a commutative diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x_0"] \arrow[r, "0"'] & \mathbb{N} \arrow[r, "S"'] & \mathbb{N} \end{tikzcd}$$

in $\mathcal{D}$. Because $(\mathbb{N}, 0, S)$ is a hnno there is a section $a : \mathbb{N} \to X$ of p such that $a0 \simeq_N x_0$ and $aS \simeq_{\mathbb{N}} fa$. By Corollary 2.57 we lift the section a to a section a' of p'. By fibrewise homotopy faithfulness (Lemma 2.49) we have that $a'0' \simeq_{\mathbb{N}'} x'_0$ because

$$F(a'0') = F(a')F(0') = a0 \simeq_N x_0 = F(x'_0)$$

and a similar argument shows $a'S' \simeq_{\mathbb{N}} f'a'$. □

Corollary 4.4. *Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence of path categories, and suppose that $\mathcal{D}$ has a hnno. Then $\mathcal{C}$ also has a hnno.*

Proof. We factorize F as PW with $P : \mathcal{E} \to \mathcal{D}$ an acyclic fibration and $W : \mathcal{C} \to \mathcal{E}$ a section of some acyclic fibration $G : \mathcal{E} \to \mathcal{C}$. By surjectivity and fullness of P we can obtain an hnno in $\mathcal{E}$ which is then preserved by G. □

By a similar argument we can conclude that weak equivalences preserve and reflect hnnos.

4.2 Homotopy exponentials

Homotopy exponentials and homotopy Π-types play the role of a homotopical version of function spaces in type theory. The homotopy Π-types take dependent types into account and need additional machinery. The definitions of homotopy exponentials and Π-types we use here are due to Den Besten and can be found in [16].

The universal properties are described in terms of properties of certain functors. We call a functor F *essentially injective* if for any two objects X and Y we have that X and Y are isomorphic if FX and FY are. We will be concerned with the following three types of functors:

e.s. essentially surjective,

e.s.e.i. essentially surjective and essentially injective,

e.s.f. essentially surjective and full.

Before we give the definition of homotopy exponentials we need a few additional results on the groupoid enrichment of path categories as introduced in Section 3.7. Recall that for a path category $\mathcal{C}$ and two objects Z and X in $\mathcal{C}$, the groupoid $\mathcal{C}(Z, X)$ has as objects the arrows $Z \to X$ and as morphisms the homotopies between the arrows identified up to fibrewise homotopy over $X \times X$. Given a function $f : X \to Y$ there is an induced functor $f \star - : \mathcal{C}(Z, X) \to \mathcal{C}(Z, Y)$. On objects it maps a morphism $g : Z \to X$ to $fg : Z \to Y$. On equivalence classes of homotopies it is defined as follows: a homotopy H is composed with the lift l arising in the diagram

$$\begin{array}{ccc} X & \xrightarrow{rf} & PY \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ PX & \xrightarrow[(fs,ft)]{} & Y \times Y \end{array}$$

to obtain a homotopy $lH : Z \to PY$. This operation is well defined. Without further ado we introduce our three types of homotopy exponentials.

Definition 4.5. Let X and Y two objects in a path category $\mathcal{C}$. A *weak / ordinary / strong homotopy exponential* consists of an object Y^X and a map $\epsilon_Y : Y^X \times X \to Y$ such that for every object T the composition

$$\mathcal{C}(T, Y^X) \xrightarrow{-\times X} \mathcal{C}(T \times X, Y^X \times X) \xrightarrow{\epsilon_Y \star -} \mathcal{C}(T \times X, Y)$$

is e.s. / e.s.e.i / e.s.f.

Remark 4.6. We remark that if $F : \mathcal{C} \to \mathcal{D}$ is an acyclic fibration of path categories and X and Y are objects in F, the functor $F_{X,Y}$ is essentially surjective and full by Lemma 4.7. Also note that two maps $f, g : X \to Y$, i.e. two objects in the groupoid $\mathcal{C}(X, Y)$, are isomorphic if they are homotopic. Together with the observation that the functor F is homotopy faithful this implies that $F_{X,Y}$ is essentially injective too.

In the proofs we will use the following facts about e.s. and e.s.f. functors.

Lemma 4.7. *Let A, B and C be groupoids and $F : A \to B$ and $G : B \to C$ functors. The following implications hold:*

- *If F and G are e.s.(f.), then GF is.*

◇ *If F and GF are e.s.(f.), then G is.*

Proof. Lemma 4.1 in [16]. □

Lemma 4.8. *Acyclic fibrations preserve homotopy exponentials.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let X, Y two objects in $\mathcal{C}$ such that its exponential Y^X exists. Let T' any object in $\mathcal{D}$. By surjectivity write $T' = F(T)$. Consider the following diagram in **Gpd**, the category of groupoids:

$$\begin{tikzcd}
\mathcal{C}(T, Y^X) \arrow[r] \arrow[d, "F_{T,Y^X}"'] & \mathcal{C}(T \times X, Y^X \times X) \arrow[r] \arrow[d, "F_{T\times X, Y^X \times X}"] & \mathcal{C}(T \times X, Y) \arrow[d, "F_{T\times X,Y}"] \\
\mathcal{D}(F(T), F(Y^X)) \arrow[r] & \mathcal{D}(F(T) \times F(X), F(Y^X) \times F(X)) \arrow[r] & \mathcal{D}(F(T) \times F(X), F(Y)).
\end{tikzcd}$$

We will show that if the upper maps compose to an e.s. / e.s.e.i / e.s.f. map, the lower does. First, we note that by the 2-categorical nature of the functor $M : \mathbf{Pth} \to \mathbf{Gpd}$, as discussed in [16], the diagram commutes. Next, we note that by Remark 4.6 the vertical maps are all e.s.(f.). By Lemma 4.7 this means that the lower part is e.s.(f.) if the upper is.

It remains to show that the lower part is e.i. if the upper part is. Suppose $f, g : F(T) \to F(Y^X)$ are such that $F(\epsilon_Y)(f \times id)$ and $F(\epsilon_Y)(g \times id)$ are isomorphic in $\mathcal{D}(F(T) \times F(X), F(Y))$. In particular, this means that $F(\epsilon_Y)(f \times id) \simeq F(\epsilon_Y)(g \times id)$ in $\mathcal{D}$. By fullness of F there are maps $f', g' : T \to Y^X$ such that $F(f') = f$ and $F(g') = g$. By the commutativity of the diagram we have that

$$F(\epsilon_Y(f' \times id)) = F(\epsilon_Y)(f \times id)$$

and

$$F(\epsilon_Y(g' \times id)) = F(\epsilon_Y)(g \times id).$$

Because $F_{T\times X,Y}$ is e.i. by Remark 4.6, the maps $\epsilon_Y(f' \times id)$ and $\epsilon_Y(g' \times id)$ are homotopic. By the fact that the upper part is e.i. it follows that f' and g' are homotopic and hence f and g are homotopic. □

Lemma 4.9. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let X and Y objects in $\mathcal{C}$. If $F(X)^{F(Y)}$ and $\epsilon_{F(Y)}$ is any type of homotopy exponential, and X^Y and ϵ_Y are mapped to respectively $F(X)^{F(Y)}$ and $\epsilon_{F(Y)}$, then X^Y and ϵ_Y form a homotopy exponential in $\mathcal{C}$*

Proof. For any object T in $\mathcal{C}$ we consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C}(T, Y^X) \arrow[r] \arrow[d, "F_{T,Y^X}"'] & \mathcal{C}(T \times X, Y^X \times X) \arrow[r] \arrow[d, "F_{T\times X, Y^X \times X}"] & \mathcal{C}(T \times X, Y) \arrow[d, "F_{T\times X,Y}"] \\
\mathcal{D}(F(T), F(Y^X)) \arrow[r] & \mathcal{D}(F(T) \times F(X), F(Y^X) \times F(X)) \arrow[r] & \mathcal{D}(F(T) \times F(X), F(Y)).
\end{tikzcd}$$

Note that we can identify $F(Y^X)$ with $F(Y)^{F(X)}$. We will show that the upper part of the diagram is respectively e.s., f. or e.i if the lower part is. This will imply that X^Y and ϵ_Y form a weak / ordinary / strong homotopy exponential if $F(X)^{F(Y)}$ and $\epsilon_{F(Y)}$ does.

Essentially surjective Let $f : T \times X \to Y$ any map in $\mathcal{C}$. Because the lower part of the diagram is essentially surjective there is a map $f' : F(T) \to F(Y^X)$ such that $F(\epsilon_Y)(f' \times id)$ is isomorphic to $F(f)$ in $\mathcal{D}(F(T) \times F(X), F(Y))$, i.e. the maps $F(\epsilon_Y)(f' \times id)$ and $F(f)$ are homotopic as maps $F(T) \times F(X) \to F(Y)$ in $\mathcal{D}$. Because F is full there exists some map $f'' : T \to Y^X$ such that $F(f'') = f'$. The diagram commutes and hence we have that

$$F(\epsilon_Y(f'' \times id)) \simeq F(f)$$

in $\mathcal{D}$. Since F is homotopy faithful, or $F_{T\times X,Y}$ is essentially injective, it follows that $\epsilon_Y(f'' \times id)$ is homotopic to f.

Full Let $f, g : T \to Y^X$ parallel morphisms in $\mathcal{C}$ and let $H : T \times X \to PY$ a representative of a morphism between $\epsilon_Y(f \times id)$ and $\epsilon_Y(g \times id)$ in $\mathcal{C}(T \times X, Y)$, i.e. a homotopy between them. Now $F(H)$ is a morphism in $\mathcal{D}(F(T) \times F(X), F(Y))$. Because the diagram commutes we again have

$$F(\epsilon_Y(f \times id)) = \epsilon_{F(Y)}(F(f) \times id)$$

and

$$F(\epsilon_Y)(g \times id)) = \epsilon_{F(Y)}(F(g) \times id).$$

By fullness of the bottom part of the diagram there is a homotopy between $F(f)$ and $F(g)$ which is mapped to a homotopy between $F(\epsilon_Y(f \times id))$ and $F(\epsilon_Y(g \times id))$ fibrewise homotopic to $F(H)$. By surjectivity on objects and fullness of F there is some $H'' : T \to P(Y^X)$ such that $F(H'') = H'$. The upper part of the diagram maps H'' to something which is then mapped by $F_{T\times X,Y}$ to something which is fibrewise homotopic to $F(H)$. Because $F_{T\times X,Y}$ is faithful we conclude that the upper part is full.

Essentially injective Suppose $f, g : T \to Y^X$ are mapped to homotopic maps $\epsilon_Y(f \times id)$ and $\epsilon_Y(g \times id)$. By commutativity of the diagram, identifying $F(\epsilon_Y)$ with $\epsilon_{F(Y)}$, and exactness of F it follows that $\epsilon_{F(Y)}(F(f) \times id)$ and $\epsilon_{F(Y)}(F(g) \times id)$ are homotopic. By essentially injectivity of the bottom part of the diagram we conclude that $F(f)$ and $F(g)$ are isomorphic in $\mathcal{D}(F(T), F(Y^X))$, i.e. $F(f) \simeq F(g)$. Because F is homotopy faithful we conclude that $f \simeq g$.

□

Corollary 4.10. *Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence of path categories, and suppose that $\mathcal{D}$ has all homotopy exponentials. Then $\mathcal{C}$ has all homotopy exponentials.*

Proof. Identical to the proof of Corollary 4.4. □

Moreover, weak equivalences preserve and reflect homotopy exponentials.

4.3 Homotopy Π-types

To define homotopy Π-types we need to work with the category $\mathcal{C}_{/X}$, which is not necessarily a path category as not every object is fibrant. However, given two objects $g : Z \to X$ and $f : Y \to X$, we can still endow $\mathcal{C}_{/X}(g, f)$ with a groupoid structure if the codomain is fibrant.

Definition 4.11. Let $f : X \to I$ and $g : I \to J$ be fibrations in $\mathcal{C}$. *A weak / ordinary / strong homotopy* Π*-type* of f and g consists of a fibration $\Pi_g f : \Pi_g X \to J$ and a map $\epsilon_X : (\Pi_g X) \times_J I \to X$ over I such that for every map $h : T \to J$ the composition

$$\mathcal{C}_{/J}(h, \Pi_g f) \xrightarrow{-\times_J I} \mathcal{C}_{/I}(h^* f, h^*\Pi_g f) \xrightarrow{\epsilon_X \star -} \mathcal{C}_{/I}(h^* f, f)$$

is respectively e.s., e.s.e.i or e.s.f.

Lemma 4.12. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let $f : X \to J$ any map and $g : Y \to J$ a fibration. The induced map*

$$F_{f,g} : \mathcal{C}_{/J}(f, g) \to \mathcal{D}_{/F(J)}(F(f), F(g))$$

is essentially surjective, essentially injective and full.

Proof. We show all three properties.

Essentially surjective This follows from Lemma 2.56.

Full Let $h, k : X \to Y$ two maps over J. Let $H : F(X) \to F(P_I Y)$ a fibrewise homotopy between $F(h)$ and $F(k)$. By Lemma 2.56 we can lift the fibrewise homotopy to $H' : X \to P_I Y$ which is a fibrewise homotopy between h and k.

Essentially injective Acyclic fibrations are fibrewise homotopy faithful by Lemma 2.49.

□

Lemma 4.13. *Acyclic fibrations preserve homotopy* Π*-types.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Given fibrations $f : X \to I$ and $g : I \to J$ in $\mathcal{C}$ such that its Π-type exists, and given any $h' : T' \to F(J)$ we get the following commutative diagram by lifting h' to be a map $h : T \to J$:

$$\begin{array}{ccccc}
\mathcal{C}_{/J}(h, \Pi_g f) & \longrightarrow & \mathcal{C}_{/I}(h^* f, h^*(\Pi_g f)) & \longrightarrow & \mathcal{C}_{/I}(h^* f, f) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{D}_{/F(J)}(F(h), F(\Pi_g f)) & \longrightarrow & \mathcal{D}_{/I}(F(h^* f), F(h^*(\Pi_g f))) & \longrightarrow & \mathcal{D}_{/F(I)}(F(h^* f), f).
\end{array}$$

We need to show that the lower composition is e.s. / e.s.e.i / e.s.f if the upper is. However, this part of the proof consists of arguments which are identical to the ones used in the proof of Lemma 4.8. □

Lemma 4.14. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let $f : X \to I$ and $g : I \to J$ fibrations in $\mathcal{C}$. Let $\Pi_{F(g)}F(f) : \Pi_{F(g)}F(X) \to F(J)$ and $\epsilon_{F(X)} : (\Pi_{F(g)}F(X)) \times_{F(J)} F(I) \to F(X)$ a weak / ordinary / strong homotopy Π-type in $\mathcal{D}$. Let $\Pi_g f : \Pi_g X \to J$ a fibration mapped to $\Pi_{F(g)}F(f)$ and let $\epsilon_X : \Pi_g X \times_J I \to X$ such that $F(\epsilon_X) = \epsilon_{F(X)}$. Then $\Pi_g f$ and ϵ_X form a weak / ordinary / strong homotopy Π-type in $\mathcal{C}$.*

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{/I}(h, \Pi_g f) \arrow[r] \arrow[d] & \mathcal{C}_{/I}(h^*f, h^*(\Pi_g f)) \arrow[r] \arrow[d] & C_{/I}(h^*f, f) \arrow[d] \\
\mathcal{D}_{/F(J)}(F(h), F(\Pi_g f)) \arrow[r] & \mathcal{D}_{/F(I)}(F(h^*f), F(h^*(\Pi_g f))) \arrow[r] & \mathcal{D}_{/F(I)}(F(h^*f), F(f)),
\end{tikzcd}$$

where we identify $F(\Pi_g f)$ with $\Pi_{F(g)}F(f)$. We need to show that the upper composition is e.s. / e.s.e.i. / e.s.f. if the lower is. However, these proofs are all identical to the ones in Lemma 4.9. □

Corollary 4.15. *Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence of path categories, and suppose that $\mathcal{D}$ has all homotopy Π-types. Then $\mathcal{C}$ has all homotopy Π-types.*

Proof. Identical to the proof of Corollary 4.4. □

Moreover, weak equivalences preserve and reflect homotopy Π-types.

4.4 Homotopical universes

Universes in type theory are types whose terms are types, closed under certain properties. In this section we will introduce the notion of a homotopical universe in a path category due to Van den Berg in [38], and show that they are closed under weak equivalences of path categories. To do this we need some additional terminology.

Definition 4.16. Let

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "p"] \\
B \arrow[r, "g"'] & Y
\end{tikzcd}$$

a commutative diagram with p a fibration. The diagram is called a *homotopy pullback* if the map $\langle f, a \rangle : A \to B \times_Y X$ is a weak equivalence.

Definition 4.17. Let $p : E \to U$ a fibration. A fibration $f : A \to B$ is called *p-small* if it arises as a homotopy pullback of p. The fibration p is a *homotopical universe* if the p-small maps are closed under composition, homotopy Π-types, and it contains all isomorphisms.

Lemma 4.18. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Let $p : A \to B$ a fibration in $\mathcal{D}$ and let $p' : A' \to B'$ mapped to p. Then F reflects p-small maps. I.e., if a map $f' : A' \to B'$ is mapped to a p-small fibration $f = Ff'$, then f' is p'-small.*

Proof. Let f' and p' as above. There is a homotopy pullback square

$$\begin{array}{ccc} A & \xrightarrow{\gamma_f} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\chi_f]{} & U. \end{array}$$

Use fullness of F to obtain the map $\chi_{f'} : B' \to U'$ such that $F(\chi_{f'}) = \chi_f$. Then one can use fullness to obtain $\langle f, \gamma_f\rangle' : A' \to B' \times_{U'} E'$ such that $F(\langle f, \gamma_f\rangle') = \langle f, \gamma_f\rangle$. Now because $F(\pi_{B'}\langle f, \gamma_f\rangle') = f$, the triangle

$$\begin{tikzcd} & B' \times_{U'} E' \arrow[d, "\pi_{B'}"] \\ A' \arrow[ur, "\langle f,\gamma_f\rangle'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

commutes up to homotopy by homotopy faithfulness of F. Applying Proposition 2.23 yields a map $\widetilde{\langle f, \gamma_f\rangle}$ satisfying $\pi_{B'}\widetilde{\langle f, \gamma_f\rangle} = f'$. Write γ'_Y for $\pi_{E'}\widetilde{\langle f, \gamma_f\rangle}$ and note that $\langle f', \gamma_{f'}\rangle = \widetilde{\langle f, \gamma_f\rangle}$. We obtain a commuting diagram

$$\begin{array}{ccc} A' & \xrightarrow{\gamma'_Y} & E' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle p'} \\ B' & \xrightarrow[\chi_{f'}]{} & U'. \end{array}$$

The induced map $\langle f', \gamma'_Y\rangle$ is homotopic to the map $\langle f, \gamma_f\rangle'$ and hence a weak equivalence, because $F(\langle f, \gamma_f\rangle' = \langle f, \gamma_f\rangle$ and F reflects weak equivalences. We conclude that f' is p'-small. $\square$

It is clear that an acyclic fibration preserves small maps relative to a fibration, since it preserves homotopy pullbacks. In fact, every exact functor preserves small maprs relative to a fibration.

Lemma 4.19. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Then F preserves homotopical universes.*

Proof. Let $p : X \to Y$ a homotopical universe in $\mathcal{C}$. We show that the $F(p)$-small maps are closed under homotopy Π-types, and leave the proofs of the other properties for the reader. Let f and g fibrations and $\Pi_g f$ its homotopy Π-type. Because F reflects homotopy Π-types we can construct the homotopy Π-type $\Pi_{g'} f'$ such that $F(\Pi_{g'} f') = \Pi_g f$ and $Ff' = f$ and $Fg' = g$. Because F reflects $F(p)$-small maps, f' and g' are p-small and hence $\Pi'_g f'$ is p-small because p is a homotopical universe. The functor F clearly preserves p-small maps and hence $\Pi_{g'} f'$ is $F(p)$-small. $\square$

Lemma 4.20. *Let $F : \mathcal{C} \to \mathcal{D}$ an acyclic fibration. Then F reflects homotopical universes in the following sense.*

Proof. Suppose a fibration $p : X \to Y$ is mapped to a homotopical universe $F(p) : F(X) \to F(Y)$. We will show that p is a homotopical universe. In Lemma 4.18 we have shown that $F(p)$-small maps are reflected. By this it follows immediately that the p-small maps are closed under composition and Π-types. We will give the prove that p-small maps are closed under composition. Let $f : A \to B$ and $g : B \to C$ two p-small maps. Then $F(f)$ and $F(g)$ are $F(p)$-small and hence $F(g)F(f)$ are. Because F reflects $F(p)$-small maps it follows that gf is p-small. $\square$

Corollary 4.21. *Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence of path categories, and suppose that $\mathcal{D}$ has a homotopical universe. Then $\mathcal{C}$ has a homotopical universe.*

Proof. Identical to the proof of Corollary 4.4. $\square$

Moreover, weak equivalences preserve and reflect homotopical universes.

4.5 Univalence

Univalence plays an important role in homotopy type theory and is moreover the axiom by which canonicity is lost, urging a notion of homotopy canonicity to arise. We will give a definition of a univalent fibration in a path category, which is introduced by Van den Berg in [38]. The definition relies on the following lemma, which can be found in the same paper.

Lemma 4.22. *Let $p : X \to Y$ a fibration, and let $f, g : Z \to Y$ two homotopic maps. Then the homotopy $H : Z \to PY$ induces a weak equivalence between f^*p and g^*p in $(\mathcal{C}_{/Z})_f$.*

Proof. We have the two pullbacks

$$\begin{array}{ccc} Z \times_Y^f X & \xrightarrow{\pi_X^f} & X \\ {\scriptstyle \pi_Z^f}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} Z \times_Y^g X & \xrightarrow{\pi_X^g} & X \\ {\scriptstyle \pi_Z^g}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[g]{} & Y. \end{array}$$

Let Γ a transport structure on p. We define a map $Z \times_Y^f X \to Z \times_Y^g X$ by

$$\langle \pi_Z^f, \Gamma\langle \pi_X^f, H\pi_Z^f \rangle\rangle.$$

Let us show that this map exists. First we show that $\langle \pi_X^f, H\pi_Z^f \rangle : Z \times_Y^f X \to X \times_Y PY$ exists. This holds true because

$$p\pi_X^f = f\pi_Z^f = sH\pi_Z^f$$

and hence by the universal property a map to the pullback is induced. Now the map $\langle \pi_Z^f, \Gamma\langle \pi_X^f, H\pi_Z^f \rangle\rangle$ exists too because

$$g\pi_Z^f = tH\pi_Z^f = t\pi_2\langle \pi_X^f, H\pi_Z^f \rangle = p\Gamma\langle \pi_X^f, H\pi_Z^f \rangle,$$

where π_2 is as in Definition 2.20. That the map is a morphism over Z is clear as

$$\pi^g_Z\langle\pi^f_Z, \Gamma\langle\pi^f_X, H\pi^f_Z\rangle\rangle = \pi^f_Z.$$

We now want to show that it is a weak equivalence. By composing H with the lift l in the following diagram:

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "r"'] & PX \arrow[d, "(t,s)"] \\ PX \arrow[r, "(s,t)"'] & X \times X \end{tikzcd}$$

we obtain a homotopy between g and f, inducing a morphism

$$\langle\pi^g_Z, \Gamma\langle\pi^g_X, lH\pi^g_Z\rangle\rangle : Z \times^g_Y X \to Z \times^f_Y X$$

in $(C_{/Z})_{\mathrm{f}}$. We will now show that the composition of these maps is homotopic to the identity. I.e. we want

$$\langle\pi^g_Z, \Gamma\langle\pi^g_X, lH\pi^g_Z\rangle\rangle\langle\pi^f_Z, \Gamma\langle\pi^f_X, H\pi^f_Z\rangle\rangle$$

to be homotopic to the identity. By Lemma 2.18 it satisfies to show that

$$\pi^f_X \simeq_Y \Gamma\langle\Gamma\langle\pi^f_X, H\pi^f_Z\rangle, lH\pi^f_Z\rangle.$$

By properties of transport (Lemma A.6 in [7]) the right hand side is fibrewise homotopic to

$$\Gamma\langle\pi^f_X, \mu(H\pi^f_Z, lH\pi^f_Z)\rangle,$$

where μ is as in Lemma A.2 in [7]. Because $\mu(1, l)$ is fibrewise homotopic to the identity we get that we are fibrewise homotopic to the map $\Gamma\langle\pi^f_X, H\pi^f_z\rangle$ Because all maps coinciding on composition with an acyclic fibration are fibrewise homotopic over it, and applying Lemma 2.18 again, we get that $\langle\pi^f_X, H\pi^f_Z\rangle$ is fibrewise homotopic to $\langle\pi^f_X, rp\pi^f_X\rangle$ and hence we get

$$\Gamma\langle\pi^f_X, H\pi^f_Z\rangle \simeq_Y \Gamma\langle\pi^f_X, rp\pi^f_X\rangle = \Gamma\langle id, rp\rangle\pi^f_X \simeq_Y \pi^f_X$$

concluding the proof. □

A fibration is univalent if every weak equivalence between such two pullbacks can be obtained by the procedure as in Lemma 4.22:

Definition 4.23. Let $\mathcal{C}$ a path category. A fibration $p : X \to Y$ is called *univalent* if for every pair of maps $f, g : Z \to Y$ and a weak equivalence $w : f^*p \to g^*p$ over Z, there is a homotopy H between f and g inducing w.

Lemma 4.24. *Let $p : X \to Y$ a fibration. Let $f, g : Z \to X$ two maps which are homotopic. Let H, K homotopies between f and g which are fibrewise homotopic over $Y \times Y$. Then the induced maps $f^*p \to g^*p$ are fibrewise homotopic over Z.*

Proof. The maps H and K are fibrewise homotopic, hence the maps $H\pi_Z^f$ and $K\pi_Z^f$ are. By Lemma 2.18 the maps $\langle \pi_Y^f, H\pi_Z^f \rangle$ and $\langle \pi_Y^f, K\pi_Z^f \rangle$ are. Because Γ preserves fibrewise homotopies the maps $\Gamma\langle \pi_Y, H\pi_Z^f \rangle$ and $\Gamma\langle \pi_Y, K\pi_Z^f \rangle$ are. By Lemma 2.18 again this suffices to show that the induced maps between the pullbacks are fibrewise homotopic. $\square$

Lemma 4.25. *Acyclic fibrations of path categories preserve univalent fibrations.*

Proof. Let $p : X \to Y$ in $\mathcal{C}$ a univalent fibration. Let $f, g : Z \to FY$ two maps in $\mathcal{D}$ such that f^*p and g^*p are connected by some weak equivalence w. We use fullness and surjectivity of F to obtain $f', g' : Z' \to X$. We use fullness of $F_{/Z} : (\mathcal{C}_{/Z})_{\mathrm{f}} \to (\mathcal{D}_{/F(Z)})_{\mathrm{f}}$ to obtain a weak equivalence $w' : (f')^*p \to (g')^*p$ mapped to w. By univalence of p we get $H : Z' \to PX$ inducing w' up to fibrewise homotopy. It is clear that FH induces w up to fibrewise homotopy. $\square$

Lemma 4.26. *Acyclic fibrations of path categories reflect univalent fibrations.*

Proof. Let $p : X \to Y$ univalent in $\mathcal{D}$ and let $p' : X \to Y$ a fibration such that $F(p') = p$. Let $f', g' : Z' \to Y'$ maps whose pullbacks along p' are connected by a weak equivalence w'. Write $F(f') :=$ and $F(g') := g$. The weak equivalence $F(w') := w$ now connects the pullbacks f^*p and g^*p since F is exact. By univalence of p there is a homotopy $H : X \to PY$ inducing the weak equivalence w. Let us now factorize the map $(f, g) : X \to Y \times Y$ to obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{H} & PY \\ {\scriptstyle w_{fg}}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ T & \xrightarrow[(p_f, p_g)]{} & Y \times Y. \end{array}$$

Write $K : T \to PY$ for a diagonal filler. Note that by the lifting property for factorizations we lift $(p_f, p_g)w_{fg}$ to $(p_{f'}, p_{g'})w_{fg'}$. Since $F_{/Y \times Y}$ is full, we can obtain $K' : T' \to PY'$ such that $F(K') = K$. The map K' is a homotopy between $p_{f'}$ and $p_{g'}$. It is clear that $K'w_{fg'}$ now defines a homotopy between f' and g'. The homotopy $K'w_{fg'}$ defines a weak equivalence u between the pullbacks by Lemma 4.22. One can check that F preserves transport. It follows by exactness that the map $F(u)$ is the same weak equivalence as the one induced by the homotopy $Kw_{fg'}$, as $F(K') = K$ and $F(w_{fg'}) = w_{fg}$. Now since $Kw_{fg'}$ is fibrewise homotopic to H, it follows by Lemma 4.24 that the weak equivalence $F(u)$ between the pullbacks which is induced by the homotopy $Kw_{fg'}$ is fibrewise homotopic to the weak equivalence induced by H, which is w. By Lemma 2.49 it now follows that u and w' are fibrewise homotopic and hence that p' is univalent. $\square$

Corollary 4.27. *Let $F : \mathcal{C} \to \mathcal{D}$ a weak equivalence of path categories, and suppose that $\mathcal{D}$ has univalent fibration. Then $\mathcal{C}$ has a univalent fibration.*

Proof. Identical to the proof of Corollary 4.4. $\square$

Moreover, weak equivalences preserve and reflect univalent fibrations.

4.6 On homotopy canonicity

We conclude this chapter with an *informal* exposition of canonicity proofs for type theory, and we will explain how the results of the previous chapters help with a proof of homotopy canonicity for objective type theory.

Let us first sketch the proof of canonicity for intuitionistic type theory with a natural numbers type as given in [11]. We refer to Theorem 20.1 in [11] and the sections afterwards for the full proof. Write $\mathcal{T}$ for the type theory of their interest. The syntactic category $S(\mathcal{T})$ in their case forms a *topos*. Moreover, it is *initial* in the category of categories modelling $\mathcal{T}$ and functors preserving all the logical structure, the *logical functors*. This means that given another category $\mathcal{C}$ with an interpretation of $\mathcal{T}$ in $\mathcal{C}$ there exists a *unique* functor $F : S(\mathcal{T}) \to \mathcal{C}$ preserving all the structure.

An example of such a model of $\mathcal{T}$ is the category of sets. In particular the natural numbers type is modelled in **Set** by its natural numbers object $\mathbb{N}$. Let $\Gamma : S(\mathcal{T}) \to \mathbb{N}$ the global sections functor, mapping an object X in $S(\mathcal{T})$ to $\mathrm{Hom}_{S(\mathcal{T})}(1, X)$. We can now *glue* along Γ to obtain a new topos $\mathrm{GL}(\Gamma)$, the gluing category of Γ. The objects of $GL(\Gamma)$ are triples (X, A, f) with X being a set, A an object of $S(\mathcal{T})$ and f a morphism from X to $\Gamma(A)$. Morphisms between such triples (X, A, f) and (Y, B, g) are pairs of morphisms $(\alpha : X \to Y, \beta : A \to B)$ such that following square commutes:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ \Gamma(A) \arrow[r, "\Gamma(\beta)"'] & \Gamma(B). \end{tikzcd}$$

The category $GL(\Gamma)$ is again a topos, and it inherits the logical sturcture of $S(\mathcal{T})$ and **Set** such that the forgetful functor $GL(\Gamma) \to S(\mathcal{T})$ is a logical functor. Let $\mathcal{N}$ denote the natural numbers object in $S(\mathcal{T})$. Then the natural numbers object in $\mathrm{GL}(\Gamma)$ is the triple $(\mathcal{N}, \mathbb{N}, f)$ where $f : \mathbb{N} \to \mathrm{Hom}_{S(\mathcal{T})}(1, \mathcal{N})$ maps a natural number n in $\mathbb{N}$ to the map $S^n(0) : 1 \to \mathcal{N}$ in $\Gamma(\mathcal{N})$.

Because $S(\mathcal{T})$ is initial we obtain a section $G : S(\mathcal{T}) \to \mathrm{GL}(\Gamma)$ of the forgetful functor. Now let x any term of natural numbers type in $\mathcal{T}$. This induces a map $x : 1 \to \mathcal{N}$ in $S(\mathcal{T})$. The section G maps x to a morphism (α_x, β_x) in $\mathrm{GL}(\Gamma)$. Note that $\beta_x = x$. We get the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r, "\alpha_x"] \arrow[d, "id"'] & \mathbb{N} \arrow[d, "f"] \\ \mathrm{Hom}_{S(\mathcal{T})}(1,1) \arrow[r, "\Gamma(\beta_x)"'] & \mathrm{Hom}_{S(\mathcal{T})}(1, \mathcal{N}) \end{tikzcd}$$

in **Set**. Since $\mathrm{Hom}_{S(\mathcal{T})}(1, 1) = 1$ the map $\Gamma(\beta_x)$ maps the single element to $x : 1 \to \mathcal{N}$. Because the diagram commutes $x : 1 \to \mathcal{N}$ is identical to the image of the single element under the map $f\alpha_x$. Because α_x is just a natural number in $\mathbb{N}$ this means that $f\alpha_x$ maps the unique element to $S^n(0)$ for some n. We conclude that x is in canonical form and that our intuitionistic type theory $\mathcal{T}$ satisfies canonicity.

The global sections functor does not suffice for path categories as the category of sets is not a model of objective type theory. The canonical path category structure on **Set** has all morphisms as fibrations and isomorphisms as weak equivalences, it is clear that in this case Γ does not preserve weak equivalences.

The idea of Sattler and Kapulkin therefore is to replace the path category $\mathcal{C}$ by the category of frames $\mathrm{Fr}\,\mathcal{C}$, and replace the set valued global sections functor by the semisimplicial set valued global sections functor we introduced in Section 3.5. By the work we have done in this chapter, and by the fact that the evaluation functor $\mathrm{ev}_0 : \mathrm{Fr}\,\mathcal{C} \to \mathcal{C}$ is an acyclic fibration, the category of frames $\mathrm{Fr}\,\mathcal{C}$ inherits all the logical structure that $\mathcal{C}$ has. By the thesis of De Boer [25] the gluing category of an exact functor is a path category again, and many of the homotopy universal constructions have been proven to be available in the gluing category if they are in the domain and the codomain. It is plausible that this holds for every homotopy universal construction.

There are now various ways to continue the proof. It is mentioned in the slides by Sattler and Kapulkin that semisimplicial sets do not model homotopy type theory. However, as simplicial sets do model homotopy type theory as proven in [5], they solve the problem by postcomposing with the functor R as in Theorem 3.12. It is still unknown whether objective type theory can be modelled in semisimplicial sets. It seems likely that it can be interpreted in simplicial sets by similarity of objective type theory and homotopy type theory. Because the functors relating semisimplicial sets and simplicial sets are weak equivalences, semisimplicial sets do have all the same homotopy universal objects as simplicial sets. The problem, however, of modelling type theory in semisimplicial sets arises in certain syntactical aspects of modelling type theory which are outside the scope of this thesis.

Independent of whether we consider Γ or $R\Gamma$ we can now glue along this functor. By the thesis of De Boer [25] the result is again a path category. Moreover, De boer has shown that homotopy natural numbers objects are available in the gluing category if they are in the domain and codomain. It is likely that similar results can be obtained for the other constructions studied in this section.

This is where the homotopy theory ends and where syntactical arguments are needed to continue. In particular we need to obtain intiality results for the syntactic category of objective type theory in the category of path categories modelling objective type theory. Although by the nature of objective type theory it is much more likely that we have initiality *up to homotopy*. Proving syntactic results like these takes a lot of work. A small example of this syntactic hassle can be found in [26], where Shulman proves homotopy canonicity for certain truncated versions of homotopy type theory. Glancing at Chapter 4 and in particular definitions 4.1 and 4.6 can give the reader an idea of the difficulty and bookkeeping involved in these syntactic aspects. For now we conclude that the homotopy theory involved in the proof idea of Sattler and Kapulkin indeed can be perfectly translated to the case of path categories and that we have a solid foundation for a potential proof of homotopy canonicity.

5 On ∞-groupoids

Combining Theorem 3.23 and Lemma 3.25 can be interpreted as a proof to the statement that "every object in a path category is an internal ∞-groupoid", if one takes (semisimplicial) Kan complexes as models of ∞-groupoids. This perfectly fits the concept that "types are ∞-groupoids" as introduced in [4]. In this chapter we will study another notion of internal ∞-groupoid, and show that every object in a path category also is an internal ∞-groupoid with respect to this definition.

The original proof in [4] has been simplified by Bourke in [39], in which another definition, due to Maltsiniotis in [19], is used as model of ∞-groupoids. An attempt to translate the proof of Bourke to the situation of path categories has been made in [17], but this turned out to be unsuccesful. However, using techniques introduced by Henry in [40] and [18] we are able to complete the proof.

5.1 Grothendieck ∞-groupoids

In this section we will introduce the specific model of ∞-groupoid, and give references for proofs in either [17] or [39].

Definition 5.1. The *category of globes* $\mathbb{G}$ is the category freely generated by the graph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \cdots$$

with the relations $\sigma_n\sigma_{n-1} = \tau_n\sigma_{n-1}$ and $\tau_n\sigma_{n-1} = \tau_n\tau_{n-1}$.

We remark that this means that $\mathrm{Hom}_{\mathbb{G}}(n,m) = \{\sigma_{n,m}, \tau_{n,m}\}$ for $n < m$ where $\sigma_{n,m}$ and $\tau_{n,m}$ are compositions of respectively σ_is and τ_is. We call the objects in the presheaf category $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ *globular sets*, and objects in the category $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ *globular objects* in $\mathcal{C}$. Because $\mathbb{G}$ is an inverse category the category $[\mathbb{G}^{\mathrm{op}}, \mathcal{C}]_{\mathrm{f,h}}$ of fibrant, homotopical globular objects in $\mathcal{C}$ forms a path category by Proposition 2.32.

Let us unpack the definition of a fibrant homotopical globular object in $\mathcal{C}$. Such an object $X_\bullet$ consists of objects X_n for every n with boundary maps $s_n, t_n : X_{n+1} \to X_n$ which are weak equivalences. Being fibrant says that these boundary maps factorize through a fibration $p_n : X_{n+1} \to M_{n+1}X_\bullet$, where $M_{n+1}X_\bullet$ denotes the matching object of $X_\bullet$ at n. Explicitly, this matching object arises as the following pullback:

$$\begin{tikzcd}
M_{n+1}X_\bullet \arrow[r, "k_n"] \arrow[d, "q_n"'] & X_n \arrow[d, "p_{n-1}"] \\
X_n \arrow[r, "p_{n-1}"'] & M_nX_\bullet.
\end{tikzcd}$$

Example 5.2. The fibrant homotopical globular objects in a path category $\mathcal{C}$ which will play a role later in this chapter are the so called *iterated path objects.* Given an object X in $\mathcal{C}$ we will construct this globular object $X_\star$ inductively. Start by setting $X_0 := X$ and choosing $X_1 := PX$ for some path object on X, and let the boundary maps be the source and target maps of PX. Because $M_1X_\star = X_0 \times X_0$ this makes $p_0 = (s_0, t_0)$ into a fibration. Now suppose we have defined $X_\star$ up to X_n with a fibration $p_{n-1} : X_n \to M_nX_\star$. Then $M_{n+1}X_\star$ is the pullback $X_n \times_{M_nX_\star} X_n$. We now let X_{n+1} be the path object on the fibration $p_{n-1} : X_n \to M_nX_\star$ in $(\mathcal{C}_{/M_nX_\star})_{\mathrm{f}}$. In other words, we obtain a factorization

$$X_n \xrightarrow{r_n} X_{n+1} \xrightarrow{p_n} M_{n+1}X_\star$$

of the diagonal $\langle id, id \rangle : X_n \to M_{n+1}X_\star$. The boundary maps $X_{n+1} \to X_n$ are now obtained by composing p_n with the projections $M_{n+1}X_\star \to X_n$.

One might think that many of the arguments on frames in path categories can be applied to the category of homotopical fibrant globular objects in a path category. This is not the case, for we can show that the evaluation functor is not always a weak equivalence.

Lemma 5.3. *The evaluation functor* $[\mathbb{G}^{\mathrm{op}}, \mathbf{Top}]_{\mathrm{f,h}} \to \mathbf{Top}$ *is not a weak equivalence.*

However, it is true that ev_0 always is a fibration. The proof of this is routine. Before we prove Lemma 5.3 we first give the following characterizations of path categories for which the evaluation functor from homotopical fibrant diagrams of the inverse category

$$\bullet \rightrightarrows \bullet$$

is a weak equivalence. Before we state the lemma, we remark that it is routine to show that the functor $\mathrm{ev}_0 : (\mathcal{C}^{\rightrightarrows})_{\mathrm{f,h}} \to \mathcal{C}$ is a fibration of path categories.

Lemma 5.4. *The following statements are equivalent:*

(i) The functor $\mathrm{ev}_0 : (\mathcal{C}^{\rightrightarrows})_{\mathrm{f,h}} \to \mathcal{C}$ *is a weak equivalence.*

(ii) All parallel acyclic fibrations in $\mathcal{C}$ *are homotopic.*

(iii) Every automorphism which is a weak equivalence in $\mathcal{C}$ *is homotopic to the identity.*

(iv) All parallel weak equivalences in $\mathcal{C}$ *are homotopic.*

Proof. We show the implications.

$(i) \implies (ii)$. Let $f, g : X \to Y$ parallel acyclic fibrations. We factorize the map $(f, g) : X \to Y \times Y$ as a weak equivalence $w : X \to T$ followed by a fibration $(p_f, p_g) : T \to Y \times Y$. By 2-out-of-3 the maps p_f and p_g are acyclic fibrations as well. In particular this means that $(p_f, p_g) : T \to Y \times Y$ is an object in $(\mathcal{C}^{\rightrightarrows})_{f,h}$. As $\mathrm{ev}_0 : (\mathcal{C}^{\rightrightarrows})_{\mathrm{f,h}} \to \mathcal{C}$ is an acyclic fibration, it is in particular full. Let PY any path object on Y with source and target maps $(s, t) : PY \to Y \times Y$. This gives an object in $(\mathcal{C}^{\rightrightarrows})_{\mathrm{f,h}}$ as well. We can use fullness on the the identity $id : Y \to Y$ to obtain some map $l : PY \to T$ such that $p_f l = s$ and $p_g l = t$. As $s \simeq t$ we have $p_f \simeq p_g$ and in particular $f \simeq g$.

(ii) $\implies$ *(iii)*. Let $w : X \to X$ a weak equivalence. We factorize w as a weak equivalence $u : X \to T$ followed by an acyclic fibration $p : T \to X$. The map u is a section of some acyclic fibration $g : T \to X$. The acyclic fibration $p : T \to X$ has some section s. As p and g are parallel acyclic fibrations they are homotopic and in particular pu and gu are homotopic. As $pu = w$ and $gu = id$ it followed that $w \simeq id$.

(iii) $\implies$ *(iv)*. Let $w, u : X \to Y$ parallel weak equivalences. Let u^{-1} the homotopy inverse of u. That is, we have $uu^{-1} \simeq id$ and $u^{-1}u \simeq id$. However, because all automorphic weak equivalences are homotopic to the identity we also have $wu^{-1} \simeq id$ and $u^{-1}w$. This means that u^{-1} becomes an inverse of w in $\mathrm{Ho}(\mathcal{C})$ and hence u and w are in the same equivalence class and hence homotopic.

(iv) $\implies$ *(i)*. We remark that all parallel weak equivalences being homotopic implies that every object $(s_0, t_0) : X_1 \to X_0 \times X_0$ has the structure of a path object on X_0. As s_0 and t_0 are homotopic, this follows from Lemma 2.24. It is clear that ev_0 reflects weak equivalences by 2-out-of-3. We show the characterization of Lemma 2.42 holds. Let $f_0 : X_0 \to Y_0$ any fibration, such that Y_0 is part of a diagram $(s_Y, t_Y) : Y_1 \to Y_0 \times Y_0$. In particular Y_1 is a path object on Y_0. Write r_Y for the constant path map of the path object Y_1.Now let X_1 a path object on X_0 with source and target maps s_X, t_X and constant path map r_X. The diagram

$$\begin{tikzcd} X_0 \arrow[r, "r_Y f_0"] \arrow[d, "r_X"'] & Y_1 \arrow[d, "(s_Y, t_Y)"] \\ X_1 \arrow[r, "(f_0 \times f_0)(s_X, t_X)"'] & Y_0 \times Y_0 \end{tikzcd}$$

has a diagonal filler f_1 which lifts f_0 to a morphism in $(\mathcal{C}^{\rightrightarrows})_{\mathrm{f,h}}$. $\square$

Proof of Lemma 5.3. Let $\mathcal{C}$ any path category. We show that $\mathrm{ev}_0 : [\mathbb{G}^{\mathrm{op}}, \mathcal{C}]_{\mathrm{f,h}} \to \mathcal{C}$ being a weak equivalence implies the properties in Lemma 5.4. Let $f, g : Y \to X$ parallel weak equivalences. We obtain a homotopical globular object $\widetilde{X_\bullet}$ by setting $\widetilde{X_0} := X$, and $\widetilde{X_n} := Y$ for $n \geq 1$. For the face maps we set $(s_0, t_0) := (f, g)$, and $(s_n, t_n) := (id, id)$ for $n > 1$. The diagram $\widetilde{X_\bullet}$ is not at all fibrant. However, we can factorize $\widetilde{X_\bullet} \to 1$ as a weak equivalence $w_\bullet : \widetilde{X_\bullet} \to X_\bullet$ followed by a fibration $X_\bullet \to 1$. The argument is identical to the factorization in Theorem 3.23. Write (s_n, t_n) for the boundary maps of $X_\bullet$. In particular we get that $s_0 w_1 = f$ and $t_0 w_1 = g$. Let $X_\star$ the iterated path object on X, with boundary maps $(s, t) : X_1 \to X_0$. Because ev_0 is an acyclic fibration it is in particular full, and hence there is a map $l : X_\bullet \to X_\star$ with $l_0 = id$. As we now have $sl = s_0$ and $tl = t_0$, and $s \simeq t$, we have that $s_0 \simeq t_0$ and in paticular $f \simeq g$.

However, it is not at all the case that in every path category all parallel weak equivalences are homotopic. A counterexample is given by considering the antipodal map $S^2 \to S^2$ and the identity on S^2, which are both homotopy equivalences but not homotopic. $\square$

Let us continue with the preliminaries for the definition of a Grothendieck ∞-groupoid. We will introduce the notion of a table of dimensions, which can be used to index certain limits.

Definition 5.5. A *table of dimensions* $\tilde{n}$ is a sequence of natural numbers $\tilde{n} = (n_1, \ldots, n_k)$ such that k is odd, and $n_{i-1} > n_i < n_{i+1}$ for i even.

Every such table of dimensions defines a subcategory $\mathbb{G}^{\tilde{n}}$ as follows. If $\tilde{n} = (n_1, \ldots, n_k)$, then $\mathbb{G}^{\tilde{n}}$ is the category

$$\begin{tikzcd} & n_2 \arrow[dl, "\tau"'] \arrow[dr, "\sigma"] & & n_4 \arrow[dl, "\tau"'] & \cdots & & n_{k-1} \arrow[dl, "\tau"'] \arrow[dr, "\sigma"] & \\ n_1 & & n_3 & & & n_{k-2} & & n_k. \end{tikzcd}$$

Given a globular object $A : \mathbb{G}^{\mathrm{op}} \to \mathcal{C}$, we call the limit of the diagram $A^{\tilde{n}} : (\mathbb{G}^{\mathrm{op}})^{\tilde{n}} \to \mathcal{C}$ an *A-globular product*. We write $A(\widetilde{n})$ for this limit. For a functor $D : \mathbb{G} \to \mathcal{C}$ we call the colimit of the diagram $\mathbb{G}^{\widetilde{n}} \to \mathcal{C}$ a $\mathcal{D}$*-globular sum*.

The category of presheaves on $\mathbb{G}$ is complete and cocomplete, and hence the the category $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ has all y-globular sums, where y denotes the Yoneda embedding. We write Θ_0 for the full subcategory of $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ consisting of the y-globular sums. The Yoneda embedding now factorizes as

$$\mathbb{G} \xrightarrow{D} \Theta_0 \longrightarrow [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$$

where D maps n to the limit of the table of dimensions (n). Let A be any globular object in $\mathcal{C}$ such that $\mathcal{C}$ has all A-globular products. Then by Lemma 2.1 in [39] there exists an unique extension

$$\begin{tikzcd} \Theta_0^{\mathrm{op}} \arrow[dr, "A(-)"] & \\ \mathbb{G}^{\mathrm{op}} \arrow[u, "D^{\mathrm{op}}"] \arrow[r, "A"'] & \mathcal{C} \end{tikzcd}$$

where $A(-)$ preserves globular products.

Definition 5.6. A *globular theory* consists of a category $\mathbb{T}$ for which the objects are the same as the objects in Θ_0, and a functor $J : \Theta_0^{\mathrm{op}} \to \mathbb{T}$ which is the identity on objects and preserves globular products.

Given a globular theory $J : \Theta_0^{\mathrm{op}} \to \mathbb{T}$, we write $Mod(\mathbb{T}, \mathcal{C})$ for the full subcategory of the functor category $[\mathbb{T}, \mathcal{C}]$ consisting of the globular product preserving functors. There is a forgetful functor $U : Mod(\mathbb{T}, \mathcal{C}) \to [\mathbb{G}^{\mathrm{op}}, \mathcal{C}]$ mapping $X : \mathbb{T} \to \mathcal{C}$ to $X \circ J \circ \mathcal{D}^{\mathrm{op}} : \mathbb{G}^{\mathrm{op}} \to \mathcal{C}$.

Definition 5.7. Let A a globular object in $\mathcal{C}$ and let $\mathbb{T}$ a globular theory. We call an object X in $Mod(\mathbb{T}, \mathcal{C})$ a $\mathbb{T}$*-algebra structure* on A if $U(X) = A$.

Definition 5.8. Let A a globular object in $\mathcal{C}$. A *parallel pair of n-cells* in A is a pair of morphisms $f, g : X \to A(n)$ for some object X, such that $s_n f = s_n g$ and $t_n f = t_n g$ or $n = 0$. A *lifting* for this parallel pair is a map $h : X \to A(n+1)$ such that $s_{n+1} h = f$ and $t_{n+1} h = g$.

Definition 5.9. Let A a globular object in $\mathcal{C}$. We call A *contractible* if each parallel pair has a lifting. A globular theory $J : \Theta_0^{\mathrm{op}} \to \mathbb{T}$ is *contractible* if $J \circ D^{\mathrm{op}}$ is contractible.

Definition 5.10. A *Grothendieck ∞-groupoid* is a $\mathbb{T}$-algebra for a contractible theory $J : \Theta_0^{\mathrm{op}} \to \mathbb{T}$.

Let us state our main theorem.

Theorem 5.11. *Let $\mathcal{C}$ a path category. For any object X in $\mathcal{C}$, the iterated path object $X_\star$ has the structure of a Grothendieck ∞-groupoid. That is, there is a contractible globular theory $J : \Theta_0^{\mathrm{op}} \to \mathbb{T}$, and a $\mathbb{T}$-algebra $Y : \mathbb{T} \to \mathcal{C}$ such that $Y \circ J \circ D^{\mathrm{op}} = X_\star$.*

We will now describe a canonical way to obtain a theory from a globular object A by means of its *endomorphism theory* $\mathrm{End}(A)$. Its objects are Θ_0 and we have

$$\mathrm{End}(A)(\tilde{n}, \tilde{m}) = \mathcal{C}(A(\tilde{n}), A(\tilde{m})).$$

This yields a factorization

$$\Theta_0^{\mathrm{op}} \xrightarrow{J_A} \mathrm{End}(A) \xrightarrow{K_A} \mathcal{C}$$

of $A(-)$. The functor J_A is the identity on objects and preserves globular products and hence is a globular theory on A.

The functor K_A maps a table of dimensions to its limit. Both J_A and K_A preserve globular products and hence K_A gives a $\mathrm{End}(A)$-algebra structure on A. We have the following characterization of contractible endomorphism theories.

Lemma 5.12. *Let $\mathcal{C}$ a category and A a globular object in $\mathcal{C}$ such that $\mathcal{C}$ has all A-globular products. Then $\mathrm{End}(A)$ is contractible iff every parallel pair of the form $f, g : A(\tilde{n}) \to A(m)$, with $\tilde{n}$ some table of dimensions, has a lift.*

Proof. Lemma 4.1 in [39], which is worked out explicitly in Lemma 22 in [17]. $\square$

5.2 Homotopy coslice categories

In this section we will give two alternative notions of the homotopy coslice category of a path category, one due to Lobski and one due to Henry. These categories will play an important role in the proof given in the last section that every object in a path category is an internal ∞-groupoid.

Definition 5.13. Let $\mathcal{C}$ a path category and X an object in $\mathcal{C}$. We write $X \mathbin{/\!\!/} \mathcal{C}$ for the *homotopy coslice category*. Its objects are equivalence classes of maps $X \to Y$ identified up to homotopy. A morphism between two equivalence classes $[n] : X \to Y$ and $[m] : X \to Z$ is a morphism $f : Y \to Z$ such that the following diagram commutes up to homotopy.

$$\begin{tikzcd} & X \arrow[dl, "n"'] \arrow[dr, "m"] & \\ Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

where n and m are any choice of representatives.

In [17] it is shown that this is a well defined category, independent of choice of representatives.

Let $X_\star$ an iterated path object of some object X. We can factorize $X_\star$ as follows

$$\mathbb{G}^{\mathrm{op}} \xrightarrow{R} X \mathbin{/\!\!/} \mathcal{C} \xrightarrow{U} \mathcal{C}.$$

The functor R is defined on n as the weak equivalence $r_n r_{n-1} \cdots r_0 : X_0 \to X_n$. It maps σ_n to s_n and τ_n to t_n.

Lemma 5.14. *Let X an object in a path category $\mathcal{C}$ and let $X_\star$ its iterated path object and $R : \mathbb{G}^{\mathrm{op}} \to X \mathbin{/\!\!/} \mathcal{C}$ as above. The homotopy coslice category $X \mathbin{/\!\!/} \mathcal{C}$ has all R-globular products, and $U : X \mathbin{/\!\!/} \mathcal{C} \to \mathcal{C}$ preserves them.*

Proof. Proposition 12 and Lemma 26 in [17]. We will give a small part of the argument for it will be important in the proof of Theorem 5.11. In the proof one uses induction on the length of the table of dimensions to show the existence of limits. For the base case the limit trivially exists. Now assuming that the limit exists for a table of dimensions $\widetilde{n} = (n_1, \dots, n_k)$, one adds two integers and obtains table of dimensions $\widetilde{n}^+ = (n_1, \dots, n_k, n_{k+1}, n_{k+1})$. Omitting the maps out of X, the limit $X_\star(\widetilde{n})$ now arises as the pullback

$$\begin{tikzcd}
X(\widetilde{n}^+) \arrow[rr] \arrow[d] & & X_\star(n_{k+2}) \arrow[d, "s"] \\
X_\star(\widetilde{n}) \arrow[r] & X_\star(n_k) \arrow[r, "t"'] & X_\star(n_{k+1}).
\end{tikzcd}$$

□

Definition 5.15. Let $\mathcal{C}$ a path category and X an object in $\mathcal{C}$. We write $\mathcal{C}^X$ for the *strong homotopy coslice category*. Its objects are fibrations $(p_0, p_1) : A \to B \times X$ such that p_1 is an acyclic fibration. Morphisms are pairs $(f : A \to A', g : B \to B')$ such that the square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "{(p_0,p_1)}"'] & A' \arrow[d, "{(p_0',p_1')}"] \\
B \times X \arrow[r, "g \times id"'] & B' \times X
\end{tikzcd}$$

commutes.

There is a canonical way to relate the two versions of the homotopy coslice category. Given an equivalence class $[n] : X \to Y$ in $X \mathbin{/\!\!/} \mathcal{C}$ we can obtain an object in $\mathcal{C}^X$ by factorizing the map $(id, n) : X \to X \times Y$. The fibration $P \to X \times Y$ is now an object in $\mathcal{C}^X$. Now let $(p_0, p_1) : A \to B \times X$ an object in $\mathcal{C}^X$. The map p_1 has a section s. We obtain an object in $X \mathbin{/\!\!/} \mathcal{C}$ by composing s with p_0.

Lemma 5.16. *Let $\mathcal{C}$ a path category and X an object in $\mathcal{C}$. Then $\mathcal{C}^X$ is a path category with weak equivalences pointwise, and fibrations the Reedy fibrations for the square without X.*

Proof. Consider the category $\mathcal{C}^{\bullet\to\bullet}$, consisting of fibrations in $\mathcal{C}$. Because $\bullet \to \bullet$ is an inverse category $\mathcal{C}^{\bullet\to\bullet}$ is a path category. Now consider the path category $(\mathcal{C}^{\bullet\to\bullet}{}_{/X\to 1})_{\mathrm{f}}$ of fibrations in $\mathcal{C}^{\bullet\to\bullet}$ with codomain $X \to 1$. We claim that this category is canonically isomorphic to the category consisting of fibrations of the form $A \to B \times X$ with A and B arbitrary objects in $\mathcal{C}$ and X fixed. Given such a fibration with codomain $X \to 1$:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & 1, \end{array}$$

we obtain a fibration $A \to B \times X$ by the fact that fibrations in $\mathcal{C}^{\bullet\to\bullet}$ are Reedy fibrations. Similarly, every fibration $A \to B \times X$ determines a fibration in $\mathcal{C}^{\bullet\to\bullet}$ between $A \to B$ and $X \to 1$.

It remains to show that the path category of fibrations $A \to B \times X$ is closed under the extra assumption that the component $A \to X$ is a weak equivalence too. In particular, we need to show that it is closed under taking pullbacks, and path objects exist. We show that path objects exist and leave the other part to the reader. However, let $(p_0, p_1) : A \to B \times X$ an object and let $(p_0', p_1') : A' \to B' \times X$ its path object. The constant path map (r_0, r_1) with components $r_0 : A \to A'$ and $r_1 : B \to B'$ consists of weak equivalences. Since $p_1' r_0 = p_1$ it follows that p_1' is a weak equivalence by 2-out-of-3. $\square$

Lemma 5.17. *The functor* $\mathrm{ev} : \mathcal{C}^X \to \mathcal{C}$ *mapping a fibration* $(p_0, p_1) : A \to B \times X$ *to* B *is a fibration of path categories.*

Proof. Consider the codomain functor $\mathrm{cod} : \mathcal{C}^{\bullet\to\bullet} \to \mathcal{C}$ evaluating a fibration $A \to B$ to its codomain B. It is routine to show that this functor is a fibration, and hence induces a fibration

$$\mathrm{cod}_{/X\to 1} : (\mathcal{C}^{\bullet\to\bullet}{}_{/X\to 1})_{\mathrm{f}} \to (\mathcal{C}_{/\,\mathrm{cod}(X\to 1)})_{\mathrm{f}}$$

where we can identify $(\mathcal{C}_{/\,\mathrm{cod}(X\to 1)})_{\mathrm{f}}$ with $\mathcal{C}$. The functor $\mathrm{cod}_{/X\to 1}$ is a fibration by Lemma 2.46, and hence the evaluation functor mapping $A \to B \times X$ to B is a fibration. It is routine to show that by restricting this evaluation functor to the objects in which the component $A \to X$ is a weak equivalence, the functor is still a fibration. $\square$

Note that it is evident that ev also reflects weak equivalences.

5.3 Homotopy initial objects

Definition 5.18. Let $\mathcal{C}$ a path category. We call an object X in $\mathcal{C}$ *homotopy initial* if for every diagram

$$\begin{array}{ccc} & & A \\ & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

with p a fibration, there exists a lift $l : X \to A$ making the triangle commute.

Lemma 5.19. *Let X homotopy initial. Let $f, g : X \to Y$ parallel maps and let $p : Y \to I$ a fibration such that $pf = pg$. Then $f \simeq_I g$.*

Proof. The diagram

$$\begin{tikzcd} & P_I Y \arrow[d] \\ X \arrow[r, "\langle f,g \rangle"'] & Y \times_I Y \end{tikzcd}$$

has a lift. □

In particular this shows that every pair of maps with homotopy initial domain are homotopic.

Lemma 5.20. *Let X any object in $\mathcal{C}$. The path object fibration $(s,t) : PX \to X \times X$ is homotopy initial in $\mathcal{C}^X$.*

Proof. Proposition 4.4.4 in [18]. We nevertheless include the proof, for it is a bit brief in the mentioned reference. Let us consider

$$\begin{tikzcd} PX \arrow[r, "f"] \arrow[d, "(s,t)"'] & A \arrow[d, "(p_0,p_1)"] & A' \arrow[l, "f'"'] \arrow[d, "(p'_0,p'_1)"] \\ X \times X \arrow[r, "g \times id"'] & B \times X & B' \times X \arrow[l, "g' \times id"] \end{tikzcd}$$

where (f', g') is a fibration in $\mathcal{C}^X$. We want to show the existence of a lift making the triangle commute. First remark that $f' : A' \to A$ is an acyclic fibration in $\mathcal{C}(X)$ and hence has a section $u : A \to A'$ over $\mathcal{C}(X)$. We obtain a map $uf : PX \to A'$ making the upper triangle commute. Let $r : X \to PX$ the constant path map. The map $v := p'_0 ufr : X \to B'$ makes the bottom triangle commute because

$$\begin{aligned} g'v &= g'p'_0 ufr \\ &= p_0 f' ufr \\ &= p_0 fr \\ &= gsr \\ &= g. \end{aligned}$$

However, this is does not fit into a morphism together with uf. Observe that the following square commutes

$$\begin{tikzcd} X \arrow[r, "ufr"] \arrow[d, "r"'] & A' \arrow[d, "\langle p'_0, f' \rangle"] \\ PX \arrow[r, "\langle vs, f \rangle"'] & B' \times_B A \end{tikzcd}$$

because

$$vsr = v = p'_0 ufr$$

and

$$fr = f'ufr.$$

We now obtain a lift $l : PX \to A'$ making the lower right triangle commute. Now we have

$$f'l = f$$

and

$$p_0'l = vs.$$

It follows that

$$p_1'l = p_1 f'l = p_1 f = t$$

and hence (l, v) is the lifting witnessing $PX \to X \times X$ being homotopy initial. □

5.4 Grothendieck ∞-groupoids in path categories

In this section we will prove Theorem 5.11. We will do this by showing that $X_\star$ factorizes as

$$\mathbb{G}^{\mathrm{op}} \xrightarrow{R} \mathcal{D} \xrightarrow{U} \mathcal{C}$$

for some category $\mathcal{D}$ such that $\mathcal{D}$ has all R-globular prodcuts, the functor U preserves them, and the endomorphism theory $J_R : \Theta_0^{\mathrm{op}} \to \mathrm{End}(R)$ is contractible. Because if this is the case, then we have indeed have a ∞-groupoid structure on $X_\star$ as UK_R is an $\mathrm{End}(R)$-algebra structure on $X_\star$. We will first show that the theorem holds for a simple case.

Lemma 5.21. *Let X be an homotopy initial object in $\mathcal{C}$ and let $X_\star$ the iterated path object on X. Then $X_\star$ is an internal ∞-groupoid.*

Proof. Consider the factorization:

$$\begin{tikzcd} \mathbb{G}^{\mathrm{op}} \arrow[r, "R"] \arrow[dr, "X_\star"'] & X \mathbin{/\!\!/} \mathcal{C} \arrow[d, "U"] \\ & \mathcal{C} \end{tikzcd}$$

we have shown that $X \mathbin{/\!\!/} \mathcal{C}$ has all R-globular products and U preserves them. It remains to show that $\mathrm{End}(R)$ is contractible. Consider any diagram

$$\begin{tikzcd} X_0 \arrow[r, "r_0^m"] \arrow[d, "r_{\tilde{n}}"'] & X_{m+1} \arrow[d, shift right, "s_m"'] \arrow[d, shift left, "t_m"] \\ X_{\tilde{n}} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_m \end{tikzcd}$$

such that $(s_{m-1}, t_{m-1})f = (s_{m-1}, t_{m-1})g$. There is an induced map $\langle f, g\rangle : X_{\widetilde{n}} \to M_{m+1}X_\star$, and hence by homotopy initiality a lift $l : X_0 \to X_{m+1}$ such that $\langle s_m, t_m\rangle l = \langle f, g\rangle r_{\widetilde{n}}$. This yields a diagram

$$\begin{tikzcd} X_0 \arrow[r, "l"] \arrow[d] & X_{m+1} \arrow[d] \\ X(\widetilde{n}) \arrow[r] & M_{m+1}X_\star \end{tikzcd}$$

which commutes strictly. And hence giving us a fibrewise homotopy $H : X(\widetilde{n}) \to X_{m+1}$ between f and g. $\square$

Lemma 5.22. *Let $F : \mathcal{C} \to \mathcal{D}$ a fibration of path categories. Let X in $\mathcal{C}$ and let $(FX)_\star$ an iterated path object on $F(X)$ in $[\mathbb{G}^{\mathrm{op}}, \mathcal{D}]_{\mathrm{f,h}}$. Then there is an iterated path object $X_\star$ on X in $[\mathbb{G}^{\mathrm{op}}, \mathcal{C}]_{f,h}$ such that $F(X_\star) = (FX)_\star$.*

Proof. We assume that $(FX)_\star$ is constructed as in Example 5.2. Then $(FX)_1$ can be lifted by the lifting property for factorizations. Now assume we have lifted up to degree $n-1$. The factorization of the image of the diagonal

$$F(\langle id, id\rangle) : F(X_{n-1}) \to F(X_{n-1} \times_{M_{n-1}X_\star} X_{n-1})$$

lifts to a factorization of $\langle id, id\rangle$. $\square$

We can now prove our theorem.

Proof of Theorem 5.11. Let $PX \to X \times X$ a path object on X and let $X_\star : \mathbb{G}^{\mathrm{op}} \to \mathcal{C}$ an iterated path object on X. Apply Lemma 5.22 to obtain an iterated path object $(PX \to X \times X)_\star$ in $\mathcal{C}^X$. We get a factorization

$$\mathbb{G}^{\mathrm{op}} \xrightarrow{(PX \to X \times X)_\star} \mathcal{C}^X \xrightarrow{\mathrm{ev}} \mathcal{C}$$

of $X_\star$. We now factorize the iterated path object $(PX \to X \times X)_\star$ and obtain

$$\mathbb{G}^{\mathrm{op}} \xrightarrow{R} (PX \to X \times X) /\!\!/ \mathcal{C}^X \xrightarrow{U} \mathcal{C}^X \xrightarrow{\mathrm{ev}} \mathcal{C}.$$

By Lemma 5.20 and the argument in Lemma 5.21 we have that the endomorphism theory J_R is contractible. By Lemma 5.14 the category $(PX \to X \times X) /\!\!/ \mathcal{C}^X$ has all R-globular products. The functor U preserves them because of Lemma 5.14. The functor ev also preserves them because it preserves pullbacks, and the globular products are inductively defined as pullbacks. $\square$

6 Conclusion and further research

In this thesis we have studied path categories and in particular the fibration category of path categories. We have shown that many of the homotopy theoretic components of a proof of homotopy canonicity claimed by Sattler and Kapulkin could be translated to the case of path categories. Besides these type theoretic motivated questions we have established results on enrichments of path categories and on internal ∞-groupoids.

We end with some pressing questions left unanswered in this thesis.

- $\diamond$ First and foremost there is the remaining part of the proof of homotopy canonicity for objective type theory. Can we work out the syntactical aspects of the proof as described in Section 4.6?
- $\diamond$ Is there a characterization of the acyclic fibrations in $\mathbf{Pth}$ for which sections exist? Is there some notion of "cofibrant" path categories? Can we obtain a subcategory $\widetilde{\mathbf{Pth}}$ of $\mathbf{Pth}$ such that $\widetilde{\mathbf{Pth}}$ becomes a path category itself?
- $\diamond$ In [15] it is shown that the fibration categories of semisimplicial fibration categories and semisimplicial tribes are weakly equivalent. Since path categories can be "squeezed" in between these two structures, we conjecture that both structures are also weakly equivalent to the fibration category of path categories.
- $\diamond$ It seems reasonable that the functor $M : \mathbf{Pth} \to \mathbf{GpdCat}$ is an exact functor. This follows immediately if one can show that the homotopy relation in a pullback of path categories is determined pointwise.
- $\diamond$ We can extend the notion of frames in a path category to a functor $\mathrm{Fr} : \mathbf{Pth} \to \mathbf{Pth}$ which takes a path category $\mathcal{C}$ to its category of frames. Since every category of frames is enriched over $\mathbf{ssSet}$ this functor induces a functor $\widetilde{\mathrm{Fr}} : \mathbf{Pth} \to \mathbf{Cat}(\mathbf{ssSet})$ of categories enriched over $\mathbf{ssSet}$. Often, if $\mathcal{M}$ is a symmetric monoidal category, the category $\mathbf{Cat}(\mathcal{M})$ can be endowed with a model structure. See for example [41] or [42]. Although $\mathbf{ssSet}$ does not carry a full model structure, it is plausible that $\mathbf{Cat}(\mathbf{ssSet})$ inherits its right semimodel structure. See [31] for this semimodel structure. We conjecture that taking the subcategory of fibrant objects for this semimodel structure on $\mathbf{Cat}(\mathbf{ssSet})$ makes $\mathbf{Cat}(\mathbf{ssSet})_{\mathrm{f}}$ into a path category. Moreover, the functor $\widetilde{\mathrm{Fr}} : \mathbf{Pth} \to \mathbf{Cat}(\mathbf{ssSet})_{\mathrm{f}}$ of path categories might be exact or almost exact as in the case of the functor $M : \mathbf{Pth} \to \mathbf{Gpd}$. A natural follow-up question is whether the fundamental groupoid functor $\Pi : \mathbf{ssSet}_{\mathrm{f}} \to \mathbf{Gpd}$ induces an exact functor $\mathbf{Cat}(\mathbf{ssSet})_{\mathrm{f}} \to \mathbf{GpdCat}$ of path categories. By Theorem 3.45 this would give us two naturally isomorphic functors $\Pi \circ \widetilde{\mathrm{Fr}}$ and $\mathrm{Fr} \circ M$.

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Populaire samenvatting

Intuïtionistische typentheorie is een formeel systeem dat als alternatief voor verzamelingenleer kan dienen als fundament van de wiskunde. In de typentheorie bestuderen we zogenaamde *typen* en *termen*. Een *type* kan je zien als een verzameling, en een *term* als een element van een bepaald type. Als een term x tot type X behoort schrijven we dat als $x : X$. In tegenstelling tot in de verzamelingenleer behoort iedere term tot *één en slechts één* type. Als we bijvoorbeeld kijken naar het getal 3, zien we deze in de verzamelingentheorie terug in meerdere verzamelingen: het is een element van $\mathbb{N}$ en $\mathbb{Z}$, maar ook van de verzameling van alle priemgetallen of van alle veelvouden van 3. In de typentheorie zit dit net wat anders; we kunnen een term niet los zien van zijn type. Zo zijn de termen $3 : \mathbb{N}$ en $3 : \mathbb{Z}$ daadwerkelijk verschillende termen die niet met elkaar in vergelijking te brengen zijn in het systeem van typentheorie. Een andere onderscheidende eigenschap van intuïtionistische typentheorie ten opzichte van verzamelingenleer is dat alle termen en alle typen expliciet geconstrueerd worden. De termen van type $\mathbb{N}$ bijvoorbeeld worden volledig geconstrueerd door de term 0 te introduceren, en verder een opvolgerterm $S(n) : \mathbb{N}$ te introduceren voor iedere term $n : \mathbb{N}$. Als we nu eisen dat $\mathbb{N}$ het kleinste type is dat 0 en al zijn opvolgers bevat, krijgen we de gebruikelijke notie van de natuurlijke getallen. Dit is een constructivistische manier van het definiëren van de natuurlijke getallen. Een voordeel hiervan is dat intuïtionistische typentheorie zich goed leent om computerprogramma's te schrijven die de correctheid van bewijzen kunnen testen.

Net als andere fundamenten van de wiskunde probeert typentheorie een zo groot mogelijke hoeveelheid wiskundige concepten te beschrijven. Eén van de belangrijkste gevolgen hiervan is dat men *proposities* of *wiskundige uitspraken* als typen wilt zien. De termen van een dergelijk type is dan een bewijs van de correctheid van de propositie. Een belangrijk voorbeeld hiervan is het *identiteitstype*: gegeven twee termen a en b van een type X, bestaat het identiteitstype $\mathrm{Id}_X(a,b)$ uit *bewijzen dat a en b gelijk zijn als termen van type X*. Omdat $\mathrm{Id}_X(a,b)$ zelf weer een type is, kunnen we dit itereren: gegeven twee bewijzen γ en ω dat a en b gelijk zijn construeren we een type $\mathrm{Id}_{\mathrm{Id}_X(a,b)}(\gamma,\omega)$ dat bewijzen bevat dat de bewijzen γ en ω gelijk zijn.

Er gebeurt iets magisch als men een type probeert voor te stellen als een *topologische ruimte* en een term van dat type als een *punt in de ruimte.* Voor twee termen a, b van type X kan men nu het identiteitstype $\mathrm{Id}_X(a,b)$ interpreteren als de ruimte van *paden van a naar b in X*. Gegeven twee van die paden γ en ω kunnen we $\mathrm{Id}_{\mathrm{Id}_X(a,b)}(\gamma,\omega)$ interpreteren als *homotopieën tussen de paden γ en ω*. We zien dat types op deze manier een hele rijke meetkundige structuur krijgen.

In deze scriptie kijken we naar bepaalde *modellen* van intuïtionistische typentheorie. In het bijzonder bestuderen we de implicatie van de meetkundige structuur van typen voor de modellen van typentheorie.